ARIMA épisciences

Quantitative stability estimate for inverse coefficients problem in linear elasticity

Houcine Meftahi[1,*] **and Taher Rezgui**[2]

[1]Université de Jendouba/Isikef, Avenue de l'Union du Maghreb Arabe 8189 Jendouba, Tunisie
[1,2]ENIT- Lamsin, BP 37, 1002 Tunis Le Belvédère, Tunisie

*E-mail : houcine.meftahi@enit.utm.tn

DOI : 10.46298/arima.9346
Submitted on April 14, 2022 - Published on December 14, 2023
Volume : **39** - Year : **2023**

Editors : *Mathieu Roche, Nabil Gmati, Bruce Watson, Clémentin Tayou Djamegni*

Abstract
In this article we consider the inverse problem of reconstructing the Lamé coefficients assumed to be piecewise constants from boundary measurements. We reformulate the inverse problem into a minimization one using a Kohn-Vogelius type functional. We study the stability of the parameters when the jump of the discontinuity is perturbed. Using tools of shape calculus, we give a quantitative stability result for local optimal solution.

Keywords
Lamé parameters; Inverse problem; Shape derivative; Stability analysis

I INTRODUCTION

The inverse problem of recovering Lamé parameters consists in finding the parameters (λ, μ) from the knowledge of the so-called Neumann-to-Dirichlet operator $\Lambda_{\lambda,\mu} : H^{-1/2}(\partial\Omega)^2 \to H^{1/2}(\partial\Omega)^2$. The operator $\Lambda_{\lambda,\mu}$ maps a boundary input $g \in H^{-1/2}(\partial\Omega)^2$ to the value $u|_{\partial\Omega}$ of the solution u of an elliptic problem with Neumann boundary condition g:

$$\Lambda_{\lambda,\mu}(g) = u|_{\partial\Omega}.$$

The two major questions for the inverse problem are the uniqueness and the stability of a solution λ, μ. The question of the uniqueness in the case of perfect data, i.e. when the Neumann-to-Dirichlet operator is completely known, has led to many difficult and interesting mathematical problems and an abundant literature on the topic is available.

The question of stability consists in studying the continuous dependence of the solution on the data. Stability is necessary to ensure that a variation of the given data in a sufficiently small range leads to an arbitrarily small change in the solution, which is primordial for applications.

This concept was introduced by Hadamard in 1902 in connection with the study of boundary value problems for partial differential equations [2].

There is a extended literature on stability issues for the inverse problem of Lamé parameters, in particular for interior stability estimates, which consist in proving that, given two maps $\Lambda_{\lambda,\mu}$ and $\Lambda_{\tilde{\lambda},\tilde{\mu}}$, one has

$$\max\left\{\|\lambda-\tilde{\lambda}\|_{L^\infty(\Omega)}, \|\mu-\tilde{\mu}\|_{L^\infty(\Omega)}\right\} \le \omega\left(\|\Lambda_{\lambda,\mu}-\Lambda_{\tilde{\lambda},\tilde{\mu}}\|_*\right),$$

where $\|.\|_*$ is the operator norm of $\mathcal{L}(H^{-1/2}(\Omega)^2, H^{1/2}(\Omega)^2)$ and $\omega : \mathbb{R}^+ \to \mathbb{R}$ is continuous with $\omega(t) \to 0$ as $t \to 0$. Recently a Lipschitz stability result has been proved in [14]. The proof relies in the monotonicity relation between the Lamé parameters, the Neumann-to-Dirichlet operator and the techniques of Localized potentials.

The inverse problem of reconstructing piecewise constants Lamé parameters λ, μ and their jump sets Γ simultaneously has been considered in [12] and the question of stability is not so well-studied.

In this contribution, we aim at giving qualitative properties of the stability in this case. We transform the inverse problem into a minimization one and study how stable is the reconstruction of the parameters λ, μ when Γ is known approximately. More precisely, we quantify the first-order stability properties of a local optimal solution of the minimization problem.

The stability estimates derived in [14] is obtained when the full Neumann-to-Dirichlet map is known. For our quantitative stability estimate we only have a finite number of measurements. To our best knowledge there are no results in this kind of stability for the problem under consideration.

In the field of PDE-constrained optimization, the differentiability and stability properties of optimal solutions with respect to parameters of the problem have been studied in details theoretically and numerically; see for instance [6–11]. Some of the ideas developed in these papers can be used for our stability analysis, unfortunately the well-known particular difficulties of differentiating with respect to the shape, in particular the fact that the set of shapes is not a vector space, prevents a direct transposition of the methods used in these papers. For our analysis on proving quantitative stability of local optimal solution, we make use of shape calculus in the spirit of [13] .

The paper is organized as follows. In section 2 we describe the setting of the direct problem, the inverse problem and the minimization problem. The stability analysis of a local optimal solution is performed in section 3.

II PROBLEM FORMULATION

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ and $\Gamma \in \mathcal{D}_{ad}$ with

$$\mathcal{D}_{ad} := \{\partial D \text{ where } D \text{ is open of class } C^2, D \subset \Omega, \inf_{x\in\partial D, y\in\partial\Omega} |x-y| > \varepsilon\}$$

for some $\varepsilon > 0$. Let Ω_2 be such that $\Gamma := \partial\Omega_2$ and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$. Throughout the paper we work with piecewise constant parameters

$$\lambda = \lambda_1\chi_{\Omega_1} + \lambda_2\chi_{\Omega_2}, \qquad \mu = \mu_1\chi_{\Omega_1} + \mu_2\chi_{\Omega_2}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}_+$ and χ denotes an indicator function. Let $0 < K_0 \leq \lambda_1, \mu_1 \leq K_1$(The parameters λ_1, μ_1 are given and not to be identified), then we define a set of piecewise constant parameters $\mathcal{P}_{ad}$ as :

$$\mathcal{P}_{ad} = \{(\lambda, \mu) = (\lambda_1 \chi_{\Omega_1} + \lambda_2 \chi_{\Omega_2}, \mu_1 \chi_{\Omega_1} + \mu_2 \chi_{\Omega_2}) : c_0 \leq, \lambda_2, \mu_2 \leq c_1\},$$

where $c_0, c_1 \in \mathbb{R}_+$. For a given load $g \in H^{-1/2}(\Gamma_N)^2$ the displacement u satisfies the following problem

$$\begin{cases} -\mathrm{div}(\mathbb{C}\hat{\nabla} u) = 0 & \text{in } \Omega \setminus \Gamma, \\ [\![u]\!] = 0 & \text{on } \Gamma, \\ [\![(\mathbb{C}\hat{\nabla} u)\nu]\!] = 0 & \text{on } \Gamma, \\ (\mathbb{C}\hat{\nabla} u)\nu = g & \text{on } \Gamma_N, \\ u = f & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{1}$$

where ν is the unit normal vector to the interface Γ or $\partial\Omega$ pointing outward of Ω_2 or Ω, respectively, $f \in H^{1/2}(\Gamma_N)^2$ and $[\![u]\!]$ denotes the jump of u across the interface Γ. The linearized strain tensor $\hat{\nabla} u$ and the stress tensor $\mathbb{C}\hat{\nabla} u$ are given by

$$\hat{\nabla} u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \qquad \mathbb{C}\hat{\nabla} u = \left(\sum_{k,l=1}^{2} \mathbb{C}_{ijkl} \frac{\partial u_k}{\partial x_l}\right)_{1 \leq i,j \leq 2},$$

with

$$\mathbb{C}_{ijkl} := \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

It is well-known that problem (1) has a unique solution $u \in \mathcal{V}$, where

$$\mathcal{V} := \left\{u \in H^1(\Omega)^2 : \quad u|_{\Gamma_D} = 0\right\}.$$

We assume that the parameters λ_1, μ_1 are known and we consider the inverse problem of recovering the parameters λ_2, μ_2 from the measurements (f, g). We are interested in the case where the interface Γ is known, but with some uncertainty. We aim to study how this uncertainty affects the reconstruction of λ_2^*, μ_2^*. Note that the results could be straightforwardly extended to the case of more than just two phases, thus for clarity it is better to study the case of two phases.

A typical approach to solve the inverse problem in practice is to consider the so-called *Kohn-Vogelius functional*. We add a regularization term which is required due to the ill-posedness of the problem

$$J((\lambda_2, \mu_2), \Gamma, u_n, u_d) := \int_\Omega \mathbb{C}\hat{\nabla}(u_n - u_d) : \nabla(u_n - u_d)\, dx + \eta \int_{\Omega_2} (\lambda_2^2 + \mu_2^2)\, dx,$$

where $\eta > 0$. Here u_n is the solution of the Neumann problem

$$\begin{cases} -\mathrm{div}(\mathbb{C}\hat{\nabla} u_n) = 0 & \text{in } \Omega \setminus \Gamma, \\ [\![u_n]\!] = 0 & \text{on } \Gamma, \\ [\![(\mathbb{C}\hat{\nabla} u_n)\nu]\!] = 0 & \text{on } \Gamma, \\ (\mathbb{C}\hat{\nabla} u_n)\nu = g & \text{on } \Gamma_N, \\ u_n = 0 & \text{on } \Gamma_D, \end{cases} \tag{2}$$

and u_d is the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}\hat{\nabla}u_d) = 0 & \text{in } \Omega \setminus \Gamma, \\ [\![u_d]\!] = 0 & \text{on } \Gamma, \\ [\![(\mathbb{C}\hat{\nabla}u_d)\nu]\!] = 0 & \text{on } \Gamma, \\ u_d = f & \text{on } \Gamma_N, \\ u_d = 0 & \text{on } \Gamma_D. \end{cases} \tag{3}$$

The variational formulations of the problems (2) and (3) read respectively

$$\text{Find } u \in \mathcal{V} \text{ such that } \int_\Omega \mathbb{C}\hat{\nabla}u : \nabla v \, dx - \int_{\Gamma_N} g \cdot v \, ds = 0 \quad \forall v \in H^1(\Omega), \tag{4}$$

$$\text{Find } u \in \mathcal{V} \text{ such that } \int_\Omega \mathbb{C}\hat{\nabla}u : \nabla v \, dx - \int_{\Gamma_N} (u-f)(\mathbb{C}\hat{\nabla}u)\nu v \, ds = 0 \quad \forall v \in H_0^1(\Omega)^2. \tag{5}$$

We associate with (4) the operator $E_n((\lambda_2, \mu_2), \Gamma, u_n) : [c_0, c_1]^2 \times \mathcal{D}_{ad} \times \mathcal{V} \to \mathcal{L}(H^1(\Omega), \mathbb{R})$ and with (5) the operator $E_d((\lambda_2, \mu_2), \Gamma, u_d) : [c_0, c_1]^2 \times \mathcal{D}_{ad} \times \mathcal{V} \to \mathcal{L}(H_0^1(\Omega), \mathbb{R}) = H^{-1}(\Omega)$. We also introduce

$$E((\lambda_2, \mu_2), \Gamma, u_n, u_d) := (E_n((\lambda_2, \mu_2), \Gamma, u_n), E_d((\lambda_2, \mu_2), \Gamma, u_d)) \,.$$

We are interested in the situation when the interface Γ is known and one wishes to reconstruct the coefficients λ_2, μ_2, the corresponding minimization problem is:

$$\begin{cases} \text{minimize } J((\lambda_2, \mu_2), \Gamma, u_n, u_d) := \displaystyle\int_\Omega \mathbb{C}\hat{\nabla}(u_n - u_d) : \hat{\nabla}(u_n - u_d) \, dx + \eta \int_{\Omega_2} (\lambda_2^2 + \mu_2^2) \, dx \\ \text{subject to } c_0 \leq \lambda_2, \mu_2 \leq c_1 \text{ and } E((\lambda_2, \mu_2), \Gamma, u_n, u_d) = (0,0). \end{cases} \tag{6}$$

Note an optimal solution $(\lambda_2^*, \mu_2^*) = (\lambda_2^*(\Gamma), \mu_2^*(\Gamma))$ of (6), if it exists, depends on Γ through the state equations.

Let us define the reduced functional corresponding to problem (6).

$$\mathcal{J}((\lambda_2, \mu_2), \Gamma) := J(\lambda_2, \mu_2, \Gamma, u_n, u_d)$$

We start with the following theorem.

Theorem 1:
The minimization problem (6) has at least one solution.

Proof. In this proof we only write $\mathcal{J}(\lambda_2, \mu_2)$ instead of $\mathcal{J}((\lambda_2, \mu_2), \Gamma)$ for simplicity since Γ is fixed. It is clear that $\inf \mathcal{J}(\lambda_2, \mu_2)$ is finite. Therefore there exists a minimizing sequence

$$\left(\lambda^k, \mu^k\right) = \left(\lambda_1 \chi_{\Omega_1} + \lambda_2^k \chi_{\Omega_2}, \mu_1 \chi_{\Omega_1} + \mu_2^k \chi_{\Omega_2}\right) \in \mathcal{P}_{ad}$$

such that

$$\lim_{k \to +\infty} \mathcal{J}(\lambda_2^k, \mu_2^k) = \inf_{c_0 \leq \lambda_2, \mu_2 \leq c_1} \mathcal{J}(\lambda_2, \mu_2).$$

The sequence (λ_2^k, μ_2^k) is bounded, thus there exists a subsequence still denoted (λ_2^k, μ_2^k) and some $(\lambda_2^*, \mu_2^*) \in \mathbb{R}^2$ such that

$$\lim_{k\to\infty} \left(\lambda^k, \mu^k\right) = \left(\lambda_1 \chi_{\Omega_1} + \lambda_2^* \chi_{\Omega_2}, \mu_1 \chi_{\Omega_1} + \mu_2^* \chi_{\Omega_2}\right).$$

By definition $u_n(\lambda_2^k, \mu_2^k)$ satisfies

$$\int_\Omega \mathbb{C}(\lambda_2^k, \mu_2^k) \hat{\nabla} u_n(\lambda_2^k, \mu_2^k) : \nabla v \, dx = \int_{\Gamma_N} g \cdot v \, ds \quad \forall v \in \mathcal{V}. \tag{7}$$

Taking $v = u_n(\lambda_2^k, \mu_2^k)$ in (7), we obtain

$$\int_\Omega \mathbb{C}(\lambda_2^k, \mu_2^k) \hat{\nabla} u_n(\lambda_2^k, \mu_2^n) : \nabla u_n(\lambda_2^k, \mu_2^n) \, dx = \int_{\Gamma_N} g \cdot u_n(\lambda_2^k, \mu_2^k) \, ds.$$

Using the Korn's inequality and a trace theorem for the right-hand side of the above equation, we deduce the existence of a constant $c > 0$ such that

$$\|u_n(\lambda_2^k, \mu_2^k)\|_{H^1(\Omega)^2} \leq c \|g\|_{L^2(\Gamma_N)^2}.$$

Therefore there exists a subsequence of $u_n(\lambda_2^k, \mu_2^k)$ still denoted $u_n(\lambda_2^k, \mu_2^k)$ such that

$$\lim_{k\to\infty} u_n(\lambda_2^k, \mu_2^k) = u_n^* \text{ weakly in } H^1(\Omega)^d,$$

and

$$\lim_{k\to\infty} u_n(\lambda_2^k, \mu_2^k) = u_n^* \text{ strongly in } L^2(\Omega)^d.$$

Letting n go to infinity in equation (7), we conclude that u_n^* satisfies

$$\int_\Omega \mathbb{C}(\lambda_2^*, \mu_2^*) \hat{\nabla} u_n^* : \nabla v \, dx = \int_{\Gamma_N} g \cdot v \, ds, \quad \forall v \in \mathcal{V}.$$

Due to the uniqueness of the weak limit we get $u_n(\lambda_2^*, \mu_2^*) = u_n^*$. This means that

$$\lim_{k\to\infty} u_n(\lambda_2^k, \mu_2^k) = u_n(\lambda_2^*, \mu_2^*) \text{ weakly in } H^1(\Omega)^2 \text{ and strongly in } L^2(\Omega)^2.$$

In the same way, we can prove that

$$\lim_{k\to\infty} u_d(\lambda_2^k, \mu_2^k) = u_d(\lambda_2^*, \mu_2^*) \text{ weakly in } H^1(\Omega)^2 \text{ and strongly in } L^2(\Omega)^2.$$

Using the lower semi-continuity of the H^1-norm yields

$$\mathcal{J}(\lambda_2^*, \mu_2^*) \leq \liminf_{k\to\infty} \mathcal{J}(\lambda_2^k, \mu_2^k) = \mathcal{J}(\lambda_2, \mu_2),$$

which concludes the proof. □

2.1 Elements of shape calculus

In this subsection we recall some basic facts about the perturbation of the identity method from shape optimization used to calculate the shape derivative. The reader is referred to [3, 4] for more details.

Let $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$, the space of continuous differentiable functions with compact support in Ω. Introduce the perturbation of identity

$$T_t = I + tV : \Omega \to \mathbb{R}^2. \tag{8}$$

Then there exists $\varepsilon > 0$ such that T_t is a diffeomorphism and $T_t(\Omega) = \Omega$ for $t \in [0, \varepsilon)$. We denote the perturbed interfaces

$$\Gamma_t := T_t(\Gamma).$$

For $t \in [0, \varepsilon)$, T_t is invertible. Furthermore, the Jacobian $\xi(t)$ is strictly positive

$$\forall\, t \in [0, \varepsilon), \quad \xi(t) = |\mathrm{det} DT_t| > 0, \tag{9}$$

where DT_t is the Jacobian matrix of the transformation T_t associated with the velocity field V. In the sequel, we use the notation M^{-1} for the inverse of M and M^{-*} for the transpose of its inverse. We also denote by

$$w(t) = \xi(t)|(DT_t)^{-*}\nu| \tag{10}$$

the tangential Jacobian of T_t on Γ.

Definition 1: Eulerian derivative
Suppose we are given a real valued shape function J defined on a subset D of $\mathbb{R}^2$. We say that J is *Eulerian semi-differentiable* at $\Omega \subset D$ in the direction V if the following limit exists in $\mathbb{R}$

$$DJ(\Omega)(V) := \lim_{t \searrow 0} \frac{J(T_t(\Omega)) - J(\Omega)}{t}.$$

If the map $V \longrightarrow DJ(\Omega)(V)$ is linear and continuous with respect to the topology of $\mathcal{D}^1(\Omega, \mathbb{R}^2)$, then J is said to be shape differentiable at Ω and $DJ(\Omega)(V)$ is called the shape derivative of J.

The Eulerian derivative is only a directional derivative. In this paper we also need the stronger notion of *Fréchet derivative*. To this end let $V \in \mathcal{V}_0$ and consider perturbations of identity $I + V$ where $\mathcal{V}_0$ is in a neighborhood of 0 in $\mathcal{D}^1(\Omega, \mathbb{R}^2)$ so that $T_V := I + V$ is a bi-Lipschitz homeomorphism. In what follows we will denote by

$$\Omega_V := T_V(\Omega)$$

Definition 2: Fréchet derivative
The functional $J(\Omega)$ is Fréchet-differentiable at Ω if there exists a linear and continuous functional $dJ(\Omega)$ from $\mathcal{D}^1(\Omega, \mathbb{R}^2)$ to $\mathbb{R}$ called *shape gradient* such that

$$J(\Omega_V) = J(\Omega) + dJ(\Omega)(V) + r(V),$$

where $|r(V)|/\|V\| \to 0$ as $\|V\| \to 0$.

In what follows we will mostly compute Fréchet derivatives given by Definition 2, however to obtain the expression of the shape derivative, it is convenient to simply compute the directional derivative given by Definition 1.

2.2 Inf-sup formulation

From the definition of the functional $\mathcal{J}$, and applying Green's formula once, we have

$$\mathcal{J}(\lambda_2, \mu_2) := J(\lambda_2, \mu_2, u_n, u_d) = J_0(\lambda_2, \mu_2, u_n) + J_0(\lambda_2, \mu_2, u_d) + J_1,$$

where

$$J_0(\lambda_2, \mu_2, u) = \int_\Omega \mathbb{C}(\lambda_2, \mu_2)\hat{\nabla} u : \hat{\nabla} u \, dx, \qquad J_1 = -2 \int_{\Gamma_\mathrm{N}} f \cdot g \, ds.$$

We introduce the Lagrangian functionals

$$G_n(\lambda_2, \mu_2, \varphi, \psi) = J_0(\lambda_2, \mu_2, \varphi) + \int_\Omega \mathbb{C}\hat{\nabla}\varphi : \hat{\nabla}\psi \, dx - \int_{\Gamma_\mathrm{N}} g \cdot \psi \, ds \quad \text{for all } \varphi, \psi \in \mathcal{V},$$

$$G_d(\lambda_2, \mu_2, \varphi, \psi) = J_0(\lambda_2, \mu_2, \varphi) + \int_\Omega \mathbb{C}\hat{\nabla}\varphi : \hat{\nabla}\psi \, dx + \int_{\Gamma_\mathrm{N}} (f - u_\mathrm{D}) \cdot (\mathbb{C}\hat{\nabla}\psi)\nu \, ds,$$

for all $\varphi \in \mathcal{V}, \psi \in H_0^1(\Omega; \mathbb{R}^2)$. Then, it is easy to check that

$$J_0(\lambda_2, \mu_2, u_n) = \inf_{\varphi \in \mathcal{V}} \sup_{\psi \in \mathcal{V}} G_n(\lambda_2, \mu_2, \varphi, \psi),$$

$$J_0(\lambda_2, \mu_2, u_d) = \inf_{\varphi \in \mathcal{V}} \sup_{\psi \in H_0^1(\Omega; \mathbb{R}^2)} G_d(\lambda_2, \mu_2, \varphi, \psi),$$

since

$$\sup_{\psi \in \mathcal{V}} G_n(\lambda_2, \mu_2, \varphi, \psi) = \begin{cases} J_0(\lambda_2, \mu_2, u_n) & \text{if } \varphi = u_n), \\ +\infty & \text{otherwise,} \end{cases}$$

$$\sup_{\psi \in H_0^1(\Omega; \mathbb{R}^2)} G_d(\mathbb{C}, \varphi, \psi) = \begin{cases} J_0(\lambda_2, \mu_2, u_d) & \text{if } \varphi = u_d), \\ +\infty & \text{otherwise.} \end{cases}$$

It is easily shown that the functional G_n (respectively G_d) is convex continuous with respect to φ and concave continuous with respect to ψ. Therefore, according to Ekeland and Temam [5], the functional G_n has a saddle point (u_n, v_n) if and only if (u_n, v_n) solves the following system:

$$\partial_\psi G_n(\lambda_2, \mu_2, u_n, v_n; \hat{\psi}) = 0,$$
$$\partial_\varphi G_n(\lambda_2, \mu_2, u_n, v_n; \hat{\varphi}) = 0,$$

for all $\hat{\psi} \in \mathcal{V}$ and $\hat{\varphi} \in \mathcal{V}$. This yields that G_n has a saddle point (u_n, v_d), where the state u_n is the unique solution of (2) and the adjoint state v_n is the solution of $\partial_\varphi G_n(\lambda_2, \mu_2, u_n, v_n; \hat{\varphi}) = 0$, or equivalently:

$$\begin{cases} -\mathrm{div}(\mathbb{C}\hat{\nabla} v_n) = 0 & \text{in } \Omega, \\ (\mathbb{C}\hat{\nabla} v_n)\nu = -2g & \text{on } \Gamma_\mathrm{N}, \\ v_n = 0 & \text{on } \Gamma_\mathrm{D}. \end{cases} \tag{11}$$

Similarly, the Lagrangian G_d has a unique saddle point (u_d, v_d) where the direct state u_d is the solution of the problem (3) and the adjoint state v_d is the unique solution of the following adjoint problem

$$\begin{cases} -\mathrm{div}(\mathbb{C}\hat{\nabla} v_d) = 0 & \text{in } \Omega, \\ v_d = 0 & \text{on } \Gamma_\mathrm{N}, \\ v_d = 0 & \text{on } \Gamma_\mathrm{D}. \end{cases} \tag{12}$$

Summarizing the above, we have obtained

Theorem 2:
The functionals $J_0(\lambda_2,\mu_2,u_n)$ and $J_0(\lambda_2,\mu_2,u_d)$ are given as

$$J_0(\lambda_2,\mu_2,u_n) = \inf_{\varphi\in\mathcal{V}} \sup_{\psi\in\mathcal{V}} G_n(\lambda_2,\mu_2,\varphi,\psi), \tag{13}$$

$$J_0(\lambda_2,\mu_2,u_d) = \inf_{\varphi\in\mathcal{V}} \sup_{\psi\in H_0^1(\Omega;\mathbb{R}^d)} G_d(\lambda_2,\mu_2,\varphi,\psi). \tag{14}$$

The unique saddle points for G_n and G_d are respectively given by (u_n,v_n) and (u_d,v_d), where $v_n = -2u_n$ and $v_d = 0$.

III STABILITY OF THE PARAMETERS WITH RESPECT TO THE INTERFACE

The optimality condition $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2,\mu_2,\Gamma)(\alpha,\beta) = 0$ for all $\alpha,\beta\in\mathbb{R}$ for problem (6) can be rewritten as

$$\text{Find } (\lambda_2^*(\Gamma),\mu_2^*(\Gamma)) \text{ such that } D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*(\Gamma),\mu_2^*(\Gamma),\Gamma) = (0,0). \tag{15}$$

In this paper we are interested in the stability of the optimal solution $(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V))$ of the minimization problem (6) with respect to the perturbed interface $\Gamma_V = (I+V)(\Gamma)$ for $V\in\mathcal{V}_0$, where $\mathcal{V}_0$ is a neighborhood of 0 in $\mathcal{D}^1(\Omega,\mathbb{R}^2)$.

Therefore we need to study the differentiability of the parameter-to-solution map $V\mapsto(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V))$. This is done by applying the implicit function theorem to the optimality conditions

$$D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V),\Gamma_V) = (0,0).$$

Essentially the idea is to linearize $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V),\Gamma_V) = (0,0)$ to get

$$D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\left(\partial_\Gamma\lambda_2^*(\Gamma)(V),\partial_\Gamma\mu_2^*(\Gamma)(V)\right) + \partial_\Gamma(D_{(\lambda_2,\mu_2),\Gamma}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma))(V) = (0,0),$$

which yields, when $D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)$ is invertible,

$$(\partial_\Gamma\lambda_2^*(\Gamma)(V),\partial_\Gamma\mu_2^*(\Gamma)(V)) = -\left(D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)^{-1}\partial_\Gamma\left(D_{(\lambda_2,\mu_2),\Gamma}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)(V).$$

In the following theorem, we prove the existence of $(\lambda_2^*(\Gamma),\mu_2^*(\Gamma))$ and we give an explicit formula for its variation with respect to Γ.

Theorem 3:
For given $\Gamma\in\mathcal{D}_{ad}$ suppose that there exists $(\lambda_2^*(\Gamma),\mu_2^*(\Gamma))$ such that

$$D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma) = (0,0),$$

and assume that $D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)$ is invertible. Then, there exists a neighborhood $\mathcal{W}$ of 0 in $\mathcal{D}^1(\mathbb{R}^2,\mathbb{R}^2)$, a neighborhood $\mathcal{U}$ of (λ_2^*,μ_2^*) in $\mathbb{R}^2$ and a C^1 function

$$\mathcal{W}\ni V\longmapsto(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V))\in\mathcal{U},$$

such that $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V),\Gamma_V) = (0,0)$, and

$$(\partial_\Gamma\lambda_2^*(\Gamma)(V),\partial_\Gamma\mu_2^*(\Gamma)(V)) = -\left(D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)^{-1}\partial_\Gamma\left(D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)(V). \tag{16}$$

Remark 1:
Theorem 3 says that if Γ_V is the perturbation of Γ by the transformation $I+V$, then the first order variation of the optimal solution is given by

$$(\lambda_2^*(\Gamma_V),\mu_2^*(\Gamma_V))-(\lambda_2^*(\Gamma),\mu_2^*(\Gamma)) \approx -\left(D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)^{-1}\partial_\Gamma\left(D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)\right)(V).$$

Proof. According to Theorem 5 and Theorem 6, the function $D_{(\lambda_2,\mu_2)}\mathcal{J}$ is of class C^1 and in view of our assumptions satisfy $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)=(0,0)$ and $D^2_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*,\mu_2^*,\Gamma)$ is invertible, we conclude the proof thanks to the implicit function theorem. □

3.1 First-order derivatives of the cost functional

In the following theorem, we give the derivative of the functional J with respect to the Lamé parameters (λ_2,μ_2).

Theorem 4:
The functional J is differentiable, and its derivative at (λ_2,μ_2) in the direction $(\tilde{\lambda}_2,\tilde{\mu}_2)$ is given by

$$\begin{aligned} D_{(\lambda_2,\mu_2)}J\left(\lambda_2,\mu_2,u_n,u_d\right)(\tilde{\lambda}_2,\tilde{\mu}_2) &= \int_\Omega \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}u_d : \hat{\nabla}u_d\,dx \\ &\quad - \int_\Omega \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}u_n : \hat{\nabla}u_n\,dx. \end{aligned} \tag{17}$$

Proof. Let $\lambda_2^t=\lambda_2+t\tilde{\lambda}_2,\quad \mu_2^t=\mu_2+t\tilde{\mu}_2,\quad \mathbb{C}=\mathbb{C}(\lambda_2,\mu_2),\quad \tilde{\mathbb{C}}=\mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2),\quad \mathbb{C}_t=\mathbb{C}+t\tilde{\mathbb{C}}$ and $t\in\mathbb{R}$ is sufficiently small parameter. Under hypotheses of Theorem 5.1 of R. Correa and A. Seeger [1], we have

$$D_{(\lambda_2,\mu_2)}J\left(\lambda_2,\mu_2,u_n,u_d\right)(\tilde{\lambda}_2,\tilde{\mu}_2)=\partial_t\tilde{G}_n(t,u_n,v_n)\Big|_{t=0}+\partial_t\tilde{G}_d(t,u_d,v_d)\Big|_{t=0},$$

where

$$\tilde{G}_n(t,\varphi,\psi):=G_n(\lambda_2^t,\mu_2^t,\varphi,\psi)=J_0(\lambda_2^t,\mu_2^t,\varphi)+\int_\Omega \mathbb{C}_t\hat{\nabla}\varphi:\hat{\nabla}\psi\,dx-\int_{\Gamma_{\mathrm{N}}} g\cdot\psi\,ds,$$

$$\begin{aligned}\tilde{G}_d(t,\varphi,\psi):=G_d(\lambda_2^t,\mu_2^t,\varphi,\psi)&=J_0(\lambda_2^t,\mu_2^t,\varphi)+\int_\Omega \mathbb{C}_t\hat{\nabla}\varphi:\hat{\nabla}\psi\,dx\\ &\quad+\int_{\Gamma_{\mathrm{N}}}(f-\varphi)\cdot(\mathbb{C}\hat{\nabla}\psi)\nu\,ds,\end{aligned}$$

and

$$\partial_t\tilde{G}_n(t,u_n,v_n)\Big|_{t=0}=-\int_\Omega \tilde{\mathbb{C}}\hat{\nabla}u_n:\hat{\nabla}u_n\,dx,$$

$$\partial_t\tilde{G}_d(t,u_g,v_d)\Big|_{t=0}=\int_\Omega \tilde{\mathbb{C}}\hat{\nabla}u_d:\hat{\nabla}u_d\,dx.$$

The above equations yield (17). To end the proof, we should verify the four assumptions $(H_1)-(H_4)$ of Theorem 5.1 of R. Correa and A. Seeger [1]. We introduce the sets

$$X_n(t) := \left\{ x^t \in \mathcal{V} : \sup_{y\in\mathcal{V}} \tilde{G}_n(t, x^t, y) = \inf_{x\in\mathcal{V}} \sup_{y\in\mathcal{V}} \tilde{G}_n(t, x, y) \right\},$$

$$Y_n(t) := \left\{ y^t \in \mathcal{V} : \inf_{x\in\mathcal{V}} \tilde{G}_n(t, x, y^t) = \sup_{y\in\mathcal{V}} \inf_{x\in\mathcal{V}} \tilde{G}_n(t, x, y) \right\},$$

$$X_d(t) := \left\{ x^t \in \mathcal{V} : \sup_{y\in H_0^1(\Omega;\mathbb{R}^d)} \tilde{G}_d(t, x^t, y) = \inf_{x\in\mathcal{V}} \sup_{y\in H_0^1(\Omega;\mathbb{R}^2)} \tilde{G}_d(t, x, y) \right\},$$

$$Y_d(t) := \left\{ y^t \in H_0^1(\Omega;\mathbb{R}^d) : \inf_{x\in\mathcal{V}} \tilde{G}_d(t, x, y^t) = \sup_{y\in H_0^1(\Omega;\mathbb{R}^2)} \inf_{x\in\mathcal{V}} \tilde{G}_d(t, x, y) \right\},$$

and obtain

$$\text{for all } \ t \in [0,\varepsilon] \quad S_n(t) = X_n(t) \times Y_n(t) = \{u_n(\mathbb{C}_t), v_n(\mathbb{C}_t)\} \neq \emptyset,$$

$$\text{for all } \ t \in [0,\varepsilon] \quad S_d(t) = X_d(t) \times Y_d(t) = \{u_{\mathrm{D}}(\mathbb{C}_t), v_d(\mathbb{C}_t)\} \neq \emptyset,$$

and assumption (H_1) is satisfied.

Assumption (H_2)*:* The partial derivatives $\partial_t\tilde{G}_n(t,\varphi,\psi), \partial_t\tilde{G}_d(t,\varphi,\psi)$ exist everywhere in $[0,\varepsilon)$ and the condition (H_2) is satisfied.
Assumptions (H_3) *and* (H_4)*:* We first show the boundedness of $(u_{\mathrm{N}}(\mathbb{C}_t), v_{\mathrm{N}}(\mathbb{C}_t))$. Letting $v = u_n(\mathbb{C}_t)$ in the variational equation

$$\int_\Omega \mathbb{C}_t \hat{\nabla} u_n(\mathbb{C}_t) : \hat{\nabla} v \, dx = \int_{\Gamma_{\mathrm{N}}} g \cdot v \, ds, \tag{18}$$

for all $v \in \mathcal{V}$, we obtain

$$\int_\Omega \mathbb{C}_t \hat{\nabla} u_n : \hat{\nabla} u_n \, dx \leq \|g\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^2)} \|u_n\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^2)}.$$

From Korn's inequality and the trace theorem, there exists $c > 0$, depending only on Ω such that

$$\|u_{\mathrm{n}}(\mathbb{C}_t)\|_{H^1(\Omega;\mathbb{R}^d)} \leq c\|g\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^2)},$$

which yields

$$\sup_{t\in[0,\varepsilon)} \|u_n(\mathbb{C}_t)\|_{H^1(\Omega;\mathbb{R}^2)} \leq c\|g\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^2)}.$$

We apply the same technique to the variational equation

$$\int_\Omega \mathbb{C}_t \hat{\nabla} u_d(\mathbb{C}_t) : \hat{\nabla} v \, dx + \int_{\Gamma_{\mathrm{N}}} (f - u_d(\mathbb{C}_t)) \cdot (\mathbb{C}\hat{\nabla} v)\nu \, ds, \tag{19}$$

for all $v \in \mathcal{V}$, and we are able to show that the function $u_d(\mathbb{C}_t)$ is bounded. The next step is to show the continuity with respect to t of $(u_n(\mathbb{C}_t), u_d(\mathbb{C}_t))$. Subtracting (18) at $t > 0$ and $t = 0$ and choosing $v = u_n(\mathbb{C}) - u_n(\mathbb{C}_t)$ yields

$$\int_\Omega \mathbb{C}\hat{\nabla}\left(u_n(\mathbb{C}) - u_n(\mathbb{C}_t)\right) : \hat{\nabla}\left(u_n(\mathbb{C}) - u_n(\mathbb{C}_t)\right)\, dx$$
$$= \int_\Omega \left(\mathbb{C} - \mathbb{C}_t\right) \hat{\nabla} u_n(\mathbb{C}_t) : \hat{\nabla}\left(u_n(\mathbb{C}) - u_n(\mathbb{C}_t)\right)\, dx.$$

Furthermore due to the boundedness of $u_n(\mathbb{C}_t)$, we obtain

$$\|u_n(\mathbb{C}_t) - u_n(\mathbb{C})\|_{H^1(\Omega,\mathbb{R}^2)} \leq C d_\Omega(\mathbb{C}_t, \mathbb{C}),$$

where

$$d_\Omega(\mathbb{C}_t, \mathbb{C}) := \max\left\{\|\lambda_2^t - \lambda_2\|_\infty, \|\mu_2^t - \mu_2\|_\infty\right\}.$$

Due to the strong continuity of $\mathbb{C}_t$ as a function of t, one deduces that $u_n(\mathbb{C}_t) \to u_n(\mathbb{C})$ in $H^1(\Omega;\mathbb{R}^2)$ as $t \to 0$. Concerning the continuity of $u_d(\mathbb{C}_t)$, one may show from (19) that $u_d(\mathbb{C}_t) \to u_d(\mathbb{C})$ in $H^1(\Omega;\mathbb{R}^2)$. Finally in view of the strong continuity of

$$(t,\varphi) \to \partial_t \tilde{G}_n(t,\varphi,\psi), \quad (t,\psi) \to \partial_t \tilde{G}_n(t,\varphi,\psi),$$
$$(t,\varphi) \to \partial_t \tilde{G}_d(t,\varphi,\psi), \quad (t,\psi) \to \partial_t \tilde{G}_d(t,\varphi,\psi),$$

assumptions (H_3) and (H_4) are verified. $\square$

3.2 Second-order derivatives of the cost functional

To compute the second order derivative of $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2,\mu_2,\Gamma)$, we use a Lagrangian method as in the previous section. We start by introducing the Lagrangian functionals

$$\begin{aligned} L_n(\lambda_2,\mu_2,\varphi,\psi) = &\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}\varphi : \hat{\nabla}\varphi\, dx + \int_\Omega \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla}\varphi : \hat{\nabla}\psi\, dx \\ &- \int_{\Gamma_N} g\cdot\psi\, ds \quad \text{for all } \varphi,\psi\in\mathcal{V}, \end{aligned} \tag{20}$$

$$\begin{aligned} L_d(\lambda_2,\mu_2,\varphi,\psi) = &\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}\varphi : \hat{\nabla}\varphi\, dx + \int_\Omega \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla}\varphi : \hat{\nabla}\psi\, dx \\ &+ \int_{\Gamma_N} (f-\varphi)\cdot(\mathbb{C}(\lambda_2,\mu_2)\hat{\nabla}\psi)\nu\, ds, \qquad \text{for all } \varphi\in\mathcal{V}, \psi\in H_0^1(\Omega)^2. \end{aligned} \tag{21}$$

The saddle point equations of L_n and L_d are given by

$$\partial_\psi L_n(\lambda_2,\mu_2,u_n,v_n;\hat{\psi}) = 0,$$
$$\partial_\varphi L_n(\lambda_2,\mu_2,u_n,v_n;\hat{\varphi}) = 0,$$

$$\partial_\psi L_d(\lambda_2,\mu_2,u_d,v_d;\hat{\psi}) = 0,$$
$$\partial_\varphi L_d(\lambda_2,\mu_2,u_d,v_d;\hat{\varphi}) = 0,$$

or equivalently

$$\int_{\Omega} \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla} u_n : \hat{\nabla}\hat{\psi}\, dx - \int_{\Gamma_N} g\cdot\hat{\psi}\, ds = 0, \tag{22}$$

$$\int_{\Omega} \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla} v_n : \hat{\nabla}\hat{\varphi}\, dx = -2\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla} u_n : \hat{\nabla}\varphi\, dx, \tag{23}$$

$$\int_{\Omega} \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla} u_d : \hat{\nabla}\hat{\psi}\, dx + \int_{\Gamma_N} (f-u_d)\cdot(\mathbb{C}(\lambda_2,\mu_2)\nabla\hat{\psi})\nu\, ds = 0, \tag{24}$$

$$\int_{\Omega} \mathbb{C}(\lambda_2,\mu_2)\hat{\nabla} v_d : \hat{\nabla}\hat{\varphi}\, dx = -2\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla} u_d : \hat{\nabla}\varphi\, dx. \tag{25}$$

We have the following two result

Theorem 5: second-order derivative
The function $D_{(\lambda_2,\mu_2)}J(\lambda_2,\mu_2,\Gamma)$ is differentiable with respect to (λ_2,μ_2) and

$$\begin{aligned} D^2_{(\lambda_2,\mu_2)}J\,(\lambda_2,\mu_2,\Gamma)\,(\tilde{\lambda}_2,\tilde{\mu}_2)(\hat{\lambda}_2,\hat{\mu}_2) &= \int_{\Omega_2} \mathbb{C}(\hat{\lambda}_2,\hat{\mu}_2)\hat{\nabla} u_n : \hat{\nabla} v_n\, dx \\ &+ \int_{\Omega_2} \mathbb{C}(\hat{\lambda}_2,\hat{\mu}_2)\hat{\nabla} u_d : \hat{\nabla} v_d\, dx, \end{aligned} \tag{26}$$

where u_n, v_n, u_d and v_d are solutions of (22), (23), (24) and (25) respectively.

Proof. As in the proof of Theorem 4, using the Lagrangian L_n and L_d for fixed Γ, and applying the results in the Appendix, we have

$$D^2_{(\lambda_2,\mu_2)}J\,(\lambda_2,\mu_2,u_n,u_d)\,(\tilde{\lambda}_2,\tilde{\mu}_2)(\hat{\lambda}_2,\hat{\mu}_2) = \partial_t\tilde{L}_n(t,u_n,v_n)\Big|_{t=0} + \partial_t\tilde{L}_d(t,u_d,v_d)\Big|_{t=0},$$

where

$$\tilde{L}_n(t,\varphi,\psi) := G_n(\lambda_2^t,\mu_2^t,\varphi,\psi) = \int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}\varphi : \hat{\nabla}\varphi\, dx + \int_{\Omega} \mathbb{C}_t\hat{\nabla}\varphi : \hat{\nabla}\psi\, dx - \int_{\Gamma_N} g\cdot\psi\, ds,$$

$$\begin{aligned} \tilde{L}_d(t,\varphi,\psi) := G_d(\lambda_2^t,\mu_2^t,\varphi,\psi) &= \int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2,\tilde{\mu}_2)\hat{\nabla}\varphi : \hat{\nabla}\varphi\, dx + \int_{\Omega} \mathbb{C}_t\hat{\nabla}\varphi : \hat{\nabla}\psi\, dx \\ &+ \int_{\Gamma_N} (f-\varphi)\cdot(\mathbb{C}\hat{\nabla}\psi)\nu\, ds, \end{aligned}$$

and

$$\partial_t\tilde{L}_n(t,u_n,v_n)\Big|_{t=0} = \int_{\Omega_2} \mathbb{C}(\hat{\lambda}_2,\hat{\mu}_2)\hat{\nabla} u_n : \hat{\nabla} v_n\, dx,$$

$$\partial_t\tilde{L}_d(t,u_d,v_d)\Big|_{t=0} = \int_{\Omega_2} \mathbb{C}(\hat{\lambda}_2,\hat{\mu}_2)\hat{\nabla} u_d : \hat{\nabla} v_d\, dx.$$

The hypotheses of Theorem 5.1 in [1] can be checked in a similar way as in the proof of Theorem 4. □

3.3 Shape derivative of the first-order derivatives of the cost functional

In the following theorem, we give the volume expression of the shape derivative of the first derivative of of the cost functional $\mathcal{J}$.

Theorem 6:
The functional $D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*, \mu_2^*, \Gamma)(\tilde{\lambda}_2, \tilde{\mu}_2)$ is shape shape differentibale with respect to Γ and its shape derivative in the direction V is given by:

$$
\begin{aligned}
&\partial_\Gamma \left(D_{(\lambda_2,\mu_2)}\mathcal{J}(\lambda_2^*, \mu_2^*, \Gamma)(\tilde{\lambda}_2, \tilde{\mu}_2) \right)(V) \\
&= \frac{1}{2}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[-\nabla u_n DV - DV^T \nabla u_n^T\right] : \left[\nabla u_n + \nabla u_n^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_n + \nabla u_n^T\right] : \left[\nabla u_n + \nabla u_n^T\right] \operatorname{div} V\, dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[-\nabla u_n DV - DV^T \nabla u_n^T\right] : \left[\nabla v_n + \nabla v_n^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_n + \nabla u_n^T\right] : \left[-\nabla v_n DV - DV^T \nabla v_n^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_n + \nabla u_n^T\right] : \left[\nabla v_n + \nabla v_n^T\right] \operatorname{div} V\, dx \\
&+ \frac{1}{2}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[-\nabla u_d DV - DV^T \nabla u_d^T\right] : \left[\nabla u_d + \nabla u_d^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_d + \nabla u_d^T\right] : \left[\nabla u_d + \nabla u_d^T\right] \operatorname{div} V\, dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[-\nabla u_d DV - DV^T \nabla u_d^T\right] : \left[\nabla v_d + \nabla v_d^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_d + \nabla u_d^T\right] : \left[-\nabla v_d DV - DV^T \nabla v_d^T\right] dx \\
&+ \frac{1}{4}\int_{\Omega_2} \mathbb{C}(\tilde{\lambda}_2, \tilde{\mu}_2)\left[\nabla u_d + \nabla u_d^T\right] : \left[\nabla v_d + \nabla v_d^T\right] \operatorname{div} V\, dx.
\end{aligned}
$$

Proof. Theorem 6 can be proven using Theorem 5.1 of R. Correa and A. Seeger [1]. □

IV CONSLUSION

In this paper we have considered the inverse problem of recovering piecewise constants Lamé paremeters. To solve the inverse problem, we minimized a Khon-Vogelius type functional. When the jump of parameters is perturbed, we derived a quantitative stability estimate using shape calculus tools and the implicit function theorem.

REFERENCES

[1] R. Correa and A. Seeger. "Directional derivative of a minimax function". In: *Nonlinear Anal.* 9.1 (1985), pages 13–22. ISSN: 0362-546X.

[2] J. Baumeister. *Stable solution of inverse problems*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1987, pages viii+254. ISBN: 3-528-08961-X.

[3] M. C. Delfour and J.-P. Zolésio. "Shape sensitivity analysis via min max differentiability". In: *SIAM J. Control Optim.* 26.4 (1988), pages 834–862. ISSN: 0363-0129.

[4] J. Sokołowski and J.-P. Zolésio. *Introduction to shape optimization*. Volume 16. Springer Series in Computational Mathematics. Shape sensitivity analysis. Berlin: Springer-Verlag, 1992, pages ii+250. ISBN: 3-540-54177-2.

[5] I. Ekeland and R. Temam. *Convex analysis and variational problems*. SIAM, 1999.

[6] K. Malanowski and F. Tröltzsch. "Lipschitz stability of solutions to parametric optimal control problems for parabolic equations". In: *Z. Anal. Anwendungen* 18.2 (1999), pages 469–489. ISSN: 0232-2064.

[7] K. Malanowski and F. Tröltzsch. "Lipschitz stability of solutions to parametric optimal control for elliptic equations". In: *Control Cybernet.* 29.1 (2000), pages 237–256. ISSN: 0324-8569.

[8] K. Malanowski. "Sensitivity analysis for parametric optimal control of semilinear parabolic equations". In: *J. Convex Anal.* 9.2 (2002). Special issue on optimization (Montpellier, 2000), pages 543–561. ISSN: 0944-6532.

[9] R. Griesse. "Parametric sensitivity analysis in optimal control of a reaction diffusion system. I. Solution differentiability". In: *Numer. Funct. Anal. Optim.* 25.1-2 (2004), pages 93–117. ISSN: 0163-0563.

[10] R. Griesse. "Parametric sensitivity analysis in optimal control of a reaction-diffusion system. II. Practical methods and examples". In: *Optim. Methods Softw.* 19.2 (2004), pages 217–242. ISSN: 1055-6788.

[11] K. Brandes and R. Griesse. "Quantitative stability analysis of optimal solutions in PDE-constrained optimization". In: *J. Comput. Appl. Math.* 206.2 (2007), pages 908–926. ISSN: 0377-0427.

[12] J. Hegemann, A. Cantarero, C. L. Richardson, and J. M. Teran. "An explicit update scheme for inverse parameter and interface estimation of piecewise constant coefficients in linear elliptic PDEs". In: *SIAM J. Sci. Comput.* 35.2 (2013), A1098–A1119. ISSN: 1064-8275.

[13] H. Kasumba, K. Kunisch, and A. Laurain. "A bilevel shape optimization problem for the exterior Bernoulli free boundary value problem". In: *Interfaces Free Bound.* 16.4 (2014), pages 459–487. ISSN: 1463-9963.

[14] S. Eberle, B. Harrach, H. Meftahi, and T. Rezgui. "Lipschitz stability estimate and reconstruction of Lamé parameters in linear elasticity". In: *Inverse Problems in Science and Engineering* (2020), pages 1–22.