Numerical study of some iterative solvers for acoustics in unbounded domains

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RÉSUMÉ. L'objectif de cet article est de présenter et d'étudier quelques méthodes itératives, utilisant les méthodes de décomposition de domaine pour la propagation d'ondes acoustiques harmoniques en domaine extérieur. On développe notre méthode dans le cas d'un guide infini dans une direction et celui du problème de diffraction par un obstacle. Dans les deux cas, on utilise des conditions aux limites transparentes connues, qui imposent sur une frontière fictive une condition aux limites utilisant un développement en série de Fourier. En vue de la mise en œuvre numérique, on propose un algorithme original, obtenu en appliquant la méthode des itérations successives au problème posé dans le domaine tronqué. Notre méthode sera interprétée comme une méthode de décomposition de domaines, ce qui permettra son étude de convergence. Les avantages de cette méthode résident dans la conservation de la structure creuse de la matrice éléments finis et la possiblilité de la factoriser une fois pour toutes au cours des itérations.

ABSTRACT. The aim of this paper is to study some iterative methods, based on the domain decomposition approach to solve the acoustic harmonic wave propagation in an unbounded domain. We describe how our methodology applies to semi-infinite closed guides and to acoustic scattering problems. In both cases, we use some well-known transparent boundary conditions by imposing on a fictitious boundary a boundary condition by the means of a Fourier expansion. For numerical purposes, we propose an original algorithm based on a fixed-point technique applied to the problem set in the truncated domain. We will interprate this method as a domain decomposition solver which allows to state convergence results. The improvement brought by this method is a consequence of the sparsity preservation of the finite matrix system which is decomposed only once.

MOTS-CLÉS : Conditions aux limites transparentes, décomposition de domaines, équation de Helmholtz.

KEYWORDS : Transparent boundary conditions, domain decomposition, Helmholtz equation.

1. Introduction

We consider first time-harmonic scalar wave propagation in an unbounded waveguide. The most direct and natural approach consists in using a truncation of the infinite waveguide by a fictitious boundary (denoted Σ) on which an artificial boundary condition is imposed. Different boundary conditions can be found in the literature, we shall focus on a nonstandard one, consisting in writing down explicitely the Dirichlet-to-Neuman operator T, which is made possible by a variables separation. We reduce the initial problem to an equivalent one, set in a bounded domain, with exact transparent boundary condition on the fictitious boundary Σ. This method has been already proposed for several problems (Lenoir-Tounsi [12], Bonnet [4], Bonnet-Starling [6], Bonnet-Gmati [5], Ferreira [10], Cutzach-Luneville[9], Mahé [13], Killer-Givoli [11], Oberai-Malhotra-Pinsky [15], Razafiarivelo [17]). The continuous problem is then discretized by a finite element method. The numerical handling of the operator T is not easy, because of its non-local character : the degrees of freedom on Σ are coupled. This breaks the typical sparsity of the finite element matrix. We propose here a new algorithm based on a fixed-point technique applied to the problem set in the truncated domain. Although it is can be applied as well to three-dimensional problems, we illustrate it here in the two-dimensional case. Even though we only consider a Neumann boundary condition on the boundaries of the waveguide, the method we present and the techniques we formulate are also valid for Dirichlet and Robin-type boundary conditions.

In order to run the convergence analysis of our method, we interprate it as an iterative non-overlapping subdomain method. Our work is then connected to others in the same field, see A. Bendali and Y. Boubendir ([3], [7]). The difference lies in the fact that we impose that the interface between the subdomains is of a separable shape. This allows us to explicitely compute the solution of the second problem allowing the economy of solving a numerical problem in exterior domain. Moreover, we can state the convergence of the relaxed algorithm ([7]). However, in the present work the convergence analysis is carried out only on a rectangular semi-infinite waveguide and for a circular scatterer. This allows to understand the importance of the various modes of propagation, on the convergence phenomenon.

To explore the performances of such iterative methods, we focus on the overlapping domain decomposition. We will show that this method improves the convergence of the evanescent modes, but deteriorates that of some propagative modes. A Krylov method can then be used for the inversion. Even though the algorithm has been ran in the case of the waveguide, it can be extended to the case of acoustic scattering problems.

The outline of the paper is as follows. The mathematical background is given in section 1. Section 2 deals with the new algorithm based on a fixed-point method, and its interpretation as a non-overlapping domain decomposition method. We then present a va riant of this algorithm relaxing the transmission condition, and an overlapping domain decomposition method. In section 3, the convergence proofs are detailed in the simple case of a rectangular semi-infinite waveguide and for a scattering problem, the scatterer being a disk. In the final, we discuss some numerical experiments.

1.1. The diffraction problem

The method is described for both problems of the waveguide and scattering by a rigid bounded body. For the sake of clarity we start with the semi-infinite waveguide. The case of an infinite waveguide is a straightforward extension. For the case of a scattering problem, we discuss only the specific results.

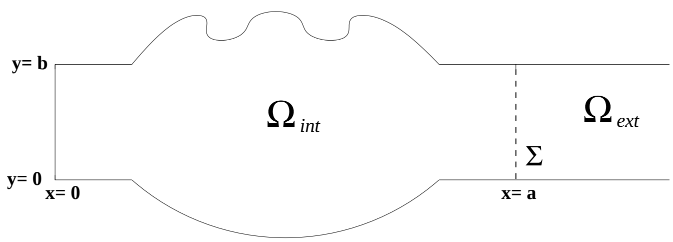

Figure 1. *Geometry of the diffraction problem*

Let a and b be two positive real numbers. We denote by Ω the bidimensional waveguide (Figure1). Then let Ω_{int} and Ω_{ext} be a partition of Ω, Ω_{int} is bounded and $\Sigma = \{(x,y) \in \mathbb{R}^2 \text{ with } x = a \text{ and } 0 < y < b\}$ is the common boundary to both subdomains. $\Omega_{ext} = \{(x,y) \in \mathbb{R}^2 \text{ with } x > a \text{ and } 0 < y < b\}$ is the reduced exterior domain as shown in Figure 1, which is a rectangular waveguide.

Let (x,y) be a generic point in Ω. The waveguide is supposed to be submitted to an incident field u_{in}. For example $u_{in} = e^{-ikx}$ is a propagative plane wave in the direction $x < 0$, where k is the wavenumber. The governing equation for the diffracted field u being the Helmholtz equation in the unbounded domain Ω, with a Neumann boundary condition on $\partial\Omega$, which needs to be completed by a radiation condition (Cutzach-Luneville [9]) in order to select the outgoing modes (Razafierivelo [17]) :

$$\left\{ \begin{array}{rcl} u_0(y) & = & O(1), \quad x \to +\infty \\ \forall m \in \mathbb{N}^*, \ \dfrac{du_m}{dx}(y) - ik_m u_m(y) & = & o(1), \quad x \to +\infty \end{array} \right. \tag{1}$$

where $u_m = (u, \varphi_m)_{0,\Sigma}$, the system $\{\varphi_m\}_{m\geq 0}$ is an orthonormal Hilbert basis of $\mathbb{L}^2(\Sigma)$, $(.,.)_{0,\Sigma}$ the inner product in $\mathbb{L}^2(\Sigma)$ and the propagation constant k_m related to the m-th mode is given by :

$$k_m = \begin{cases} (k^2 - \dfrac{m^2\pi^2}{b^2})^{\frac{1}{2}}, & m < \dfrac{kb}{\pi} \\ i(\dfrac{m^2\pi^2}{b^2} - k^2)^{\frac{1}{2}}, & m \geq \dfrac{kb}{\pi} \end{cases} \tag{2}$$

For the diffraction problem by a rigid bounded obstacle ω in $\mathbb{R}^2$, Ω denotes the complement of $\overline{\omega}$ in $\mathbb{R}^2$. Ω_{int} and Ω_{ext} is a partition of Ω. Their common boundary Σ is a circle of radius a. For this second problem the radiation condition is the well known Sommerfeld condition : $\partial u/\partial r - iku = o(1/r^{1/2}), r \to +\infty$. Let (P) be the problem set in Ω, solved by the diffracted field u.

$$(P) \begin{cases} \text{Find} \quad u \in \mathbb{H}^1_{loc}(\Omega), \text{such that} \\ \Delta u + k^2 u = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} \qquad = f \quad \text{on} \quad \partial\Omega \\ + \text{ radiation condition,} \end{cases}$$

where $\mathbb{H}^1_{loc}(\Omega)$ is the space of functions that belong to $\mathbb{H}^1(\mathcal{D})$ for any open-bounded set $\mathcal{D}$ in Ω and $f = -\dfrac{\partial u_{in}}{\partial \nu}$ arises from the incident wave u_{in}. In the waveguide case, in order to simplify the presentation, f is assumed to be compactly supported, i.e. supp $(f) \subset \partial\Omega_{int}$. It is the case of an incident plane wave $u_{in} = e^{-ikx}$, which is independant of the coordinate y. We aim a finite element approximation, then, we shall introduce next the localized finite element method, that allows to reduce the computations to a bounded domain.

1.2. A reduced problem

The localized finite element method consists in truncating the initial domain Ω to the bounded one Ω_{int} and in using a transparent boundary condition on the fictitious boundary Σ, based on the modes of the rectangular waveguide. This condition is expressed in terms of the Dirichlet-to-Neuman operator, which is made possible by variables separation.

The reduced differential problem to be solved can be stated as follows :

$$(P_\Sigma)\begin{cases} \text{Find} \quad u \in \mathbb{H}^1(\Omega_{int}), \text{such that} \\ \Delta u + k^2 u \quad = \quad 0 \quad \text{in} \quad \Omega_{int} \\ \dfrac{\partial u}{\partial \nu} \quad = \quad f \quad \text{on} \quad \partial\Omega_{int} \setminus \Sigma \\ \dfrac{\partial u}{\partial \nu} \quad = \quad T(u_{|\Sigma}) \quad \text{on} \quad \Sigma. \end{cases}$$

where T is the mapping from $\mathbb{H}^{1/2}(\Sigma)$ on $\mathbb{H}^{-1/2}(\Sigma)$ defined by :

$$T(\varphi) = \sum_{m\in\mathbb{N}} ik_m(\varphi, \varphi_m)_{0,\Sigma}\varphi_m \tag{3}$$

k_m were introduced in [2], and the orthonormal Hilbert basis $\{\varphi_m\}_{m\geq 0}$ is defined to be

$$\varphi_0(y) = \sqrt{\frac{1}{b}} \text{ and } \varphi_m(y) = \sqrt{\frac{2}{b}}\cos(\frac{m\pi}{b}y), \ \forall m \geq 1.$$

Remark 1 : *For the waveguide, the resulting problem (P_Σ) is well posed, except for at most a countable set of irregular frequencies ([17]). Moreover, it is proven in [17] that it is equivalent to (P) (defined in section 1.1). Precisely, if u is a solution of (P) then $u_{|\Omega_{int}}$ is a solution of (P_Σ). Conversely, if $\tilde{u}$ is a solution of (P_Σ), then there is a unique extension of it, that solves (P). This means that the last equation in problem (P_Σ) is a perfect nonreflecting (or absorbing) boundary condition, which prevents the waves reflection on the artificial boundary, and can thus be considered as a radiation condition. For the scattering problem, (P_Σ) is well posed and equivalent to (P), in the sense already mentioned above.*

For the scattering problem, the fictitious boundary Σ is a circle with radius a, and the Dirichlet to Neumann Operator T defined on Σ is given by :

$$T(\varphi) = \sum_{m\in\mathbb{Z}} \frac{k(H_m^{(1)})'(ka)}{H_m^{(1)}(ka)}(\varphi, \varphi_m)_{0,\Sigma}\varphi_m. \tag{4}$$

Here $H_m^{(1)}$ is the Hankel function of the first kind and $\{\varphi_m(\theta) = \dfrac{1}{\sqrt{2\pi}}e^{im\theta}\}_{m\in\mathbb{Z}}$ is a Hilbert Basis of $\mathbb{L}^2(\Sigma)$. The truncated problem can be put under the following variational form :

$$\begin{cases} \text{Find } u \in \mathbb{H}^1(\Omega_{int}), u \neq 0 \text{ such that} \\ a(u,v) - c(u,v) = l(v), \quad \forall v \in \mathbb{H}^1(\Omega_{int}), \end{cases}$$

where the forms $a(.,.)$, $c(.,.)$ and $l(.)$ are defined by :

$$\begin{aligned} a(u,v) &= \int_{\Omega_{int}} (\nabla u \nabla \overline{v} - k^2 u \overline{v})\, d\Omega_{int} \\ c(u,v) &= < Tu, v >_{-\frac{1}{2},\frac{1}{2},\Sigma} \\ l(v) &= \int_{(\partial\Omega_{int}\setminus\Sigma)} f\overline{v}\, d\gamma \end{aligned}$$

where the brackets $< .,. >_{-\frac{1}{2},\frac{1}{2},\Sigma}$ denote the duality product between $\mathbb{H}^{-1/2}(\Sigma)$ and $\mathbb{H}^{1/2}(\Sigma)$.

The continuous problem (P_Σ) is then discretized by a finite element method. We introduce the shape function w_α for each node α and $(u_\alpha$ the approximation of the solution in the node α. For numerical purposes, we use a truncation of the operator T denoted by T^M, where M is related to the number of terms. Then the linear system can be formulated as follows :

$$(A - C)U = F \tag{5}$$

In (5), we set $A = (a(w_\alpha, w_\beta))_{\alpha,\beta}$, $F = (l(w_\beta))_\beta$, $U = (u_\alpha)_\alpha$ and $C = (C_{\alpha,\beta})_{\alpha,\beta}$ such that, $\forall\ (\alpha,\beta) \in \mathbb{N}^2$:

$$C_{\alpha,\beta} = (T^M w_\alpha, w_\beta)_{0,\Sigma} = \sum_{|m|\le M} \mu_m (w_\alpha, \varphi_m)_{0,\Sigma} \overline{(w_\beta, \varphi_m)}_{0,\Sigma}$$

where μ_m is equal to ik_m for the waveguide and to $\left(k(H_m^{(1)})'(ka)\right)/H_m^{(1)}(ka)$ for the scattering problem.

Handling numerically the fictitious boundary condition may arise some trouble becomes of the non-local character. The local support of the shape functions induces a band structure of the stiffness matrix because there is no interaction between two non-neighbour nodes. However $C_{\alpha,\beta}$ is nonzero for any pair of nodes α and β related to Σ, since the degrees of freedom are coupled together. The typical sparsity of finite element matrix is broken, the method that we propose consists in using a fixed-point method for (P_Σ), in order to restore the sparsity of the obtained linear system at each iteration.

2. A fixed-point algorithm

The iterative algorithm we use and study is based on a fixed-point technique applied to the problem (P_Σ) set in the truncated domain Ω_{int}, with a transparent boundary condition on Σ.

2.1. A non-overlapping domain decomposition method

The main point is to write, at each iteration, the boundary condition as follows : $\dfrac{\partial u^{n+1}}{\partial \nu} = Tu^n$ on Σ, which changes the "T condition" to a Neumann condition. The resulting diffraction problem is not well-posed for a countable set of values of the wave-number, due to the irregular frequencies of the domain Ω_{int}. However, these singularities are not intrinsic to the original problem, which suggests to modify the boundary condition in order to overcome this difficulty. Let us introduce the following problem :

$$(P_\Sigma^{n+1}) \begin{cases} \text{Find } \ u^{n+1} \in \mathbb{H}^1(\Omega_{int}), \text{such that :} \\ \Delta u^{n+1} + k^2 u^{n+1} = 0 & \text{in} \quad \Omega_{int} \\ \dfrac{\partial u^{n+1}}{\partial \nu} = f & \text{on} \quad \partial\Omega_{int} \setminus \Sigma \\ \dfrac{\partial u^{n+1}}{\partial \nu} - iku^{n+1} = Tu^n - iku^n & \text{on} \quad \Sigma \end{cases}$$

This problem with a Robin condition on Σ is known to be well posed. By the means of a variational formulation, we show that the solution ψ of the homogenous problem associated to (P_Σ^{n+1}), satisfies $\dfrac{\partial \psi}{\partial \nu}_{|\Sigma} = \psi_{|\Sigma} = 0$. The theorem of unique continuation leads to $\psi = 0$ on the domain Ω_{int}. The algorithm is implemented under this form and can start from u^0 which satisfies $\dfrac{\partial u^0}{\partial \nu} - iku^0 = 0$, on Σ.

To carry out the convergence analysis of the proposed method, we need an interpretation of it as an iterative non-overlapping subdomains method. As we will see the presented algorithm is nothing else than solving alternatly a sequence of problems set on the subdomains Ω_{int} and Ω_{ext}. The boundary conditions are chosen iteratively by some appropriate transmission conditions between adjacent subdomains (Collino-Ghanemi-Joly [8], Benamou-Desprès [2], Boubendir [7]). The iterative procedure is defined using two sequences $(u^n)_{n\in\mathbb{N}}$ and $(v^n)_{n\in\mathbb{N}}$, solving the following problems :

$$(P_{int}^{n+1}) \begin{cases} u^{n+1} \in \mathbb{H}^1(\Omega_{int}); \\ \Delta u^{n+1} + k^2 u^{n+1} = 0 & (\Omega_{int}) \\ \dfrac{\partial u^{n+1}}{\partial \nu} = f & (\partial\Omega_{int} \setminus \Sigma) \\ \dfrac{\partial u^{n+1}}{\partial \nu} - iku^{n+1} = \dfrac{\partial v^n}{\partial \nu} - ikv^n & (\Sigma) \end{cases} \quad (P_{ext}^{n}) \begin{cases} v^n \in \mathbb{H}^1_{loc}(\Omega_{ext}), \\ (\Delta + k^2) v^n = 0 & (\Omega_{ext}) \\ \dfrac{\partial v^n}{\partial \nu} = 0 & (\partial\Omega_{ext} \setminus \Sigma) \\ v^n = u^n & (\Sigma) \\ \text{radiation condition} \end{cases}$$