An introduction to the topological asymptotic expansion with examples

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ABSTRACT. To find an optimal domain is equivalent to look for its characteristic function. At first sight this problem seems to be nondifferentiable. But it is possible to derive the variation of a cost function when we switch the characteristic function from 0 to 1 or from 1 to 0 in a small area. Classical and two generalized adjoint approaches are considered in this paper. Their domain of validity is given and illustrated by several examples. Using this gradient type information, it is possible to build fast algorithms. Generally, only one iteration is needed to find the optimal shape.

RÉSUMÉ. Trouver un domaine optimal est équivalent à la recherche de sa fonction caractéristique. A première vue, ce problème semble non différentiable, mais il est possible de calculer la variation de la fonction coût lorsque la fonction caractéristique passe de 1 à 0 ou de 0 à 1 dans une région de petite taille. On s'appuyera sur une approche adjointe classique et deux généralisations de cette méthode. Le domaine de validité de ces différentes approches est donné et illustré par différents exemples. Cette information de type gradient permet de construire des algorithmes très efficaces : en général, une seule itération suffit pour trouver le domaine optimal.

KEYWORDS : Topological sensitivity, shape optimization, inverse problem, Lagrange operator, adjoint methods

MOTS-CLÉS : Sensibilité topologique, optimisation de forme, problème inverse, opérateur de Lagrange, méthodes ajointes

1. Introduction

Most of the important contributions in topological optimization concern structural mechanics and particularly the optimization of the compliance (strain potential energy) under a volume constraint. Following the idea that the optimal structure has a lot of small holes, some authors [AlK93, Ben96] use a class of composite materials. This approach leads to homogenization theory [All02]. The field of applications of these methods is quite restricted. Global optimization techniques like genetic algorithms are used in order to solve more general problems [KaS97, SKJ96]. Unfortunately these methods are very slow.

Level-set methods give interesting results [San96, AlT02]. It consists of making the boundary of the domain evolve according to a transport equation. This allows the number of holes in the domain to decrease, but not to increase.

The topological asymptotic expansion is general and efficient. As a background, we cite the contribution of Schumacher [EsS94, Sch95] under the name of bubble method in the context of compliance optimization with a Neumann boundary condition on the unknown boundary. It consists in inserting a small hole according to topological sensitivity information, then this small hole blows following classical shape optimization methods. Let us mention recent promising results, that have been obtained by coupling level-set methods with a topological asymptotic method [BHR04, WYW04, AGJ05, AmA05].

This paper is an introduction to topological asymptotic expansion methods [Ili92, SoZ99, MNP00, AVV01, GGM01, GuS01, Mas02, AVV03, SAM03, HaM04, MPS05, AHM05, Ams05]. More exactly, a shape optimization problem consists in minimizing a functional $j(\Omega) = J(\Omega, u_\Omega)$ where u_Ω is the solution to a Partial Differential Equation defined in the domain Ω. Let us consider $\Omega_\varepsilon = \Omega \backslash B(x, \varepsilon)$ where $B(x, \varepsilon)$ is the ball of radius ε about the center x. An asymptotic expansion of the functional j can be obtained in the following form:

$$j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon) g(x) + o(f(\varepsilon))$$

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0, f(\varepsilon) > 0.$$

The function g is called the *topological gradient*.

Most of shape optimization problems could be considered via material properties optimization: we look for the distribution of a material property c taking two values c_1 and c_2. If the cost function $c \mapsto \mathcal{J}(c)$ is differentiable on L^p, $p < 2$, then the topological sensitivity g is derived easily from the classical gradient of $\mathcal{J}$, it gives the variation of $\mathcal{J}$ if we switch the material property from c_1 to c_2 or from c_2 to c_1 in a small region. In section 2, we recall the classical adjoint approach. Its basic concept is to say that the gradient of the cost function is equal to the partial derivative of the Lagrange operator with respect to c: it is not necessary to calculate the variation of the state (the solution of the direct problem) with respect to c.

Unfortunately, in many relevant cases from the practical point of view, the cost function is differentiable with respect to c, but in L^∞. To derive a topological sensitivity, we will consider two ways to generalize the adjoint approach. In both cases, it will be necessary to take into account the variation of the state with respect to topological perturbation to obtain the variation of the cost function. This variation is known [AVV01, AVV03, AmK04, MNP00], it depends on the shape of the hole and on the boundary condition on the boundary of the hole. When the hole is simple (disc, ellipse, straight crack, ...), we obtain an explicit expansion of the solution and for more general holes, this expansion could be obtained numerically.

In section 3, we consider the first generalization technique. The existence of a Frechet-like expansion of the Lagrange operator is assumed. In particular, the leading term of the expansion is continuous with respect to state and adjoint variables. This hypothesis is satisfied if we consider a domain truncation technique around the hole and a boundary condition based on a Dirichlet-to-Neumann operator. We work in a fixed domain and a fixed functional space. The variation of the Lagrange operator with respect to the size of the hole ϵ is given by the variation of the Dirichlet-to-Neumann operator. Calculating this variation requires the knowledge of the expansion of the solution associated to the Dirichlet-to-Neumann problem with respect to ϵ.

The second generalization of the adjoint technique is presented in section 4. The Lagrange operator does not admit a Frechet-like expansion. In this case, it is necessary to consider its total variation and to take into account the asymptotic expansion of the state. We give at the end of this section some applications of the topological asymptotic expansion to crack detection. The topological gradient at the first iteration gives a good localization of the cracks.

When an iterative algorithm is needed, we refer to [GGM01, GuS01, Mas02, SAM03, HaM04, Ams05] for the presentation of a fixed point method based on the work of Céa et al. [CGM73].

2. From differential calculus to 0-1 optimization

We show in this section that it is possible, under weak hypotheses, to derive topological asymptotic expansion from classical gradient.

Let us consider a bounded domain $\Omega \subset \mathbf{R}^N$ $(N = 1, 2, 3)$ and the elliptic problem

$$\begin{cases} \nabla \cdot (\alpha \nabla u) + \beta u = b & \text{in } \Omega, \\ B.C. & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where α and β are two functions defined in Ω. The functions α, β, and the boundary condition $B.C.$ will be specified so that the problem (1) and its topological perturbations admit one and only one solution.

If α goes to 0 in $\omega \subset\subset \Omega$, the corresponding solution u_α tends to the solution u_N of the problem with Neumann boundary condition:

$$\begin{cases} \nabla \cdot (\alpha \nabla u_N) + \beta u_N = b & \text{in } \Omega \setminus \omega, \\ \partial_n u_N = 0 & \text{on } \partial\omega, \\ B.C. & \text{on } \partial\Omega. \end{cases}$$

If β goes to ∞ in $\omega \subset\subset \Omega$ the corresponding solution tends to the solution u_D of the problem with Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (\alpha \nabla u_D) + \beta u_D = b & \text{in } \Omega \setminus \omega, \\ u_D = 0 & \text{on } \partial\omega, \\ B.C. & \text{on } \partial\Omega. \end{cases}$$

This second case recalls the standard penalization method used in finite elements methods for the implementation of a Dirichlet condition.

2.1. From classical gradient to topological sensitivity

Let $1 \leq p < 2$, we denote by $\mathcal{J}$ a differentiable functional on $L^p(\Omega)$

$$\begin{array}{rccc} \mathcal{J}: & L^p(\Omega) & \longrightarrow & \mathbf{R} \\ & c & \longmapsto & \mathcal{J}(c) \end{array}$$

and let $g \in L^{p'}(\Omega)$ be the Riesz representation of its differential $\mathcal{J}'(c)$. For all $\delta c \in L^p(\Omega)$ we have:

$$\mathcal{J}(c + \delta c) = \mathcal{J}(c) + \int_\Omega g(x)\delta c(x)\,dx + o(||\delta c||_p).$$

We wish to study the variation of the functional $\mathcal{J}$ with respect to a finite topological perturbation δc_ϵ. What we have in mind is a perturbation of a finite value in a region of small volume, described as follows: let κ be a real

$$\delta c_\epsilon = \begin{cases} \kappa & \text{in } B(x_0, \epsilon) \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

Hypothesis 1 *Let $1 \leq p < 2$. We make the following assumptions:*

1-a) *there exists a constant $\gamma_1 > 0$ such that for all $c \in L^p(\Omega)$ and for all $\delta c \in L^p(\Omega)$ we have:*

$$|\mathcal{J}(c + \delta c) - \mathcal{J}(c) - \mathcal{J}'(c).\delta c| \leq \gamma_1 ||\delta c||_p{}^2.$$

1-b) *the function g is Lipschitz continuous, in other words there is a constant $\gamma_2 > 0$ such that for all $x, y \in \Omega$:*

$$|g(x) - g(y)| \leq \gamma_2 ||x - y||.$$

Theorem 1 *Let δc_ϵ be defined by (2) and the cost function j defined by*

$$j(\epsilon) = \mathcal{J}(c + \delta c_\epsilon).$$

If hypothesis 1 holds, then when $\epsilon \to 0$,

$$j(\epsilon) = j(0) + \kappa\rho(\epsilon)g(x_0) + o(\rho(\epsilon)),$$

where $\rho(\epsilon) = meas(B(x_0, \epsilon))$.

The function g is called the topological gradient (of the cost function $\mathcal{J}$, relative to a jump of κ).

Proof: The perturbation δc_ϵ given by (2) is small in $L^p(\Omega)$ when $\epsilon \to 0$ since

$$||\delta c_\epsilon||_p = \kappa\rho(\epsilon)^{1/p}.$$

The right-hand side in hypothesis 1-a) is then

$$||\delta c_\epsilon||_p^2 = (\kappa\rho(\epsilon)^{1/p})^2 = \kappa^2\rho(\epsilon)^{2/p} = o(\rho(\epsilon)),$$

since $p < 2$. The derivative $\mathcal{J}'(c).\delta c_\epsilon$ can be estimated as follows using hypothesis 1-b):

$$|\mathcal{J}'(c).\delta c_\epsilon - \kappa g(x_0)\rho(\epsilon)| = \left| \int_\Omega g\delta c_\epsilon - \kappa g(x_0)\rho(\epsilon) \right| \leq \int_{B(x_0,\epsilon)} \kappa |g(x) - g(x_0)| \, dx \leq \gamma_2\kappa\epsilon\rho(\epsilon).$$

These two results give:

$$|j(\epsilon) - j(0) - \kappa\rho(\epsilon)g(x_0)| = o(\rho(\epsilon)).$$

□

2.2. The Dirichlet condition

Let us consider a C^1 bounded domain $\Omega \subset \mathbf{R}^N$, with $N = 1, 2$ or 3 and let $\mathcal{V} \subset H^1(\Omega)$ be a Hilbert space. Let a be a bilinear, continuous and coercive form on $\mathcal{V}$.

For all $c \in L^p(\Omega)$, we denote by $a_c \in \mathcal{L}_2(\mathcal{V})$ the bilinear form defined on $\mathcal{V}$ by:

$$\begin{array}{rccl} a_c : & \mathcal{V} \times \mathcal{V} & \longrightarrow & \mathbf{R} \\ & (u, v) & \longmapsto & a_c(u, v) = a(u, v) + \displaystyle\int_\Omega c\, u\, v\, dx \end{array} \qquad (3)$$

Lemma 1 *The map*

$$\begin{array}{ccc} L^p(\Omega) & \longrightarrow & \mathcal{L}_2(\mathcal{V}) \\ c & \longmapsto & a_c \end{array}$$

is continuous for

$$\left\{ \begin{array}{ll} p \geq 1 & \text{when } N = 1 \\ p > 1 & \text{when } N = 2 \\ p > 3/2 & \text{when } N = 3 \end{array} \right. \tag{4}$$

Since it is an affine map, it is then (twice) differentiable on $L^p(\Omega)$.

Proof: It follows from Rellich-Kondrachov's theorem (see e.g. [Bre83]), that $H^1(\Omega) \subset L^q(\Omega)$ for

$$\left\{ \begin{array}{ll} 1 \leq q \leq \infty & \text{when } N = 1 \\ 1 \leq q < \infty & \text{when } N = 2 \\ 1 \leq q < 6 & \text{when } N = 3 \end{array} \right. \tag{5}$$

Let q satisfy (5), and let p be such that $\dfrac{1}{p} + \dfrac{2}{q} = 1$. We have for $u, v \in \mathcal{V}$:

$$\begin{aligned} |a_c(u,v) - a_{c'}(u,v)| = \left| \int_\Omega (c - c')uv \right| &\leq ||c - c'||_p ||uv||_{q/2} \\ &\leq ||c - c'||_p \, ||u||_q \, ||v||_q \\ &\leq K ||c - c'||_p \, ||u||_{\mathcal{V}} ||v||_{\mathcal{V}}, \end{aligned}$$

where the two first inequalities follow from Hölder's inequality and the last one from the continuity of the inclusion $\mathcal{V} \subset L^q(\Omega)$.

For $N = 1$, $1 \leq q \leq \infty$ gives $p > 1$.

For $N = 2$, $1 \leq q < \infty$ gives $p \geq 1$.

For $N = 3$, $1 \leq q < 6$ gives $1 < p < 3/2$. □

Let ℓ be a linear continuous form on $\mathcal{V}$ and $u_c \in \mathcal{V}$ be the unique solution of the following variational problem:

$$a_c(u_c, v) = \ell(v) \qquad \forall v \in \mathcal{V}. \tag{6}$$

Lemma 2 *Let p satisfy (4) and J be a differentiable functional defined on $\mathcal{V}$, we consider the cost-function*

$$\begin{array}{rccl} \mathcal{J}: & L^p(\Omega) & \longrightarrow & \mathbf{R} \\ & c & \longmapsto & \mathcal{J}(c) = J(u_c). \end{array}$$

Let u_c be the direct state solution of (6), and let $p_c \in \mathcal{V}$ be the adjoint state solution of the adjoint problem

$$a_c(\psi, p_c) = -DJ(u_c).\psi \qquad \forall \psi \in \mathcal{V}. \tag{7}$$

Then the functional $\mathcal{J}$ is differentiable and for all $\delta c \in L^p(\Omega)$:

$$\mathcal{J}'(c).\delta c = \int_\Omega \delta c.u_c.p_c \, dx.$$

In other words, the Riesz representation of the differential g of $\mathcal{J}$ is

$$g = u_c p_c.$$

Proof: We use the Lagrangian method introduced in [Lio76] and [Cea86]. Let us consider the Lagrangian $\mathcal{L}$ defined on $L^p(\Omega) \times \mathcal{V} \times \mathcal{V}$ by

$$\mathcal{L}(c, u, v) = J(u) + a_c(u, v) - \ell(v).$$

The Lagrangian $\mathcal{L}$ admits partial differentials with respect to each of the 3 variables and we have:

$$D_1\mathcal{L}(c, u, v).\delta c = a_{\delta c}(u, v) - a(u, v), \qquad D_2\mathcal{L}(c, u, v).\phi = DJ(u).\phi + a_c(\phi, v).$$

Moreover, the map $L^p \to \mathcal{V},\ c \mapsto u_c$ is differentiable because of lemma 1, equation (6) and implicit functions theorem. When $v \in \mathcal{V}$ is fixed, for every $c \in L^p(\Omega)$ we have $\mathcal{J}(c) = J(u_c) = \mathcal{L}(c, u_c, v)$. It follows that for every $\delta c \in L^p(\Omega)$,

$$\mathcal{J}'(c).\delta c = D_1\mathcal{L}(c, u_c, v).\delta c + D_2\mathcal{L}(c, u_c, v).D_c(u_c).\delta c.$$

Note that if $v = p_c$ then $D_2\mathcal{L}(c, u_c, p_c) = 0$, hence

$$\mathcal{J}'(c).\delta c = D_1\mathcal{L}(c, u_c, p_c) = \int_\Omega \delta c.u_c.p_c \, dx. \qquad \square.$$

Theorem 2 *Let $p < 2$ satisfy (4). Let a_c be given by (3) and u_c the solution of (6). We assume that J is twice differentiable on $\mathcal{V}$ and consider the particular case of $c \equiv 0$ and $\delta c = \delta c_\epsilon$ defined by (2). We assume that the direct state u_0 and the adjoint state p_0 are such that $u_0 p_0$ is a Lipschitzian function.*

The cost function j defined by $j(\epsilon) = \mathcal{J}(u_{\delta c_\epsilon})$ has the following expansion:

$$j(\epsilon) - j(0) = \kappa\rho(\epsilon) u_0.p_0 + o(\rho(\epsilon)).$$

Proof : Lemma 2 proves that $\mathcal{J}$ is differentiable on $\mathcal{V}$. When looking at the proof, it appears that it is even twice differentiable, hence hypothesis 1-a) holds, and $\mathcal{J}'(0) = u_0 p_0$. Our assumptions imply that hypothesis 1-b) is also satisfied. The asymptotic expansion of j follows from theorem 1. $\square$

When the parameter κ is large, the solution $u_{\delta c_\epsilon}$ is close to zero in $B(x_0, \epsilon)$. A penalization technique provides a good approximation of the solution in $\Omega \setminus B(x_0, \epsilon)$ with a homogeneous Dirichlet condition on the boundary of the hole $B(x_0, \epsilon)$. The topological gradient $g = u_0 p_0$ is exactly what is obtained in next section using a more sophisticated approach (see also [GGM01, GuS01, HaM04, Mas02, SAM03]).

We consider now two one-dimensional examples, where the solution can be explicitly calculated. In the first example the classical gradient is equal to the topological gradient. In the second example, the classical gradient and the topological gradient are different.

2.3. First example

Consider the following one-dimensional elliptic state equation:

$$\left\{ \begin{array}{ll} -u'' + c\,u = 0 & \text{for } 0 < x < 1 \\ u(0) = 0 & \\ u'(1) = 1, & \end{array} \right. \tag{8}$$

its variational formulation is

$$\left\{ \begin{array}{l} \text{find } u \in \mathcal{V} \text{ such that} \\ \displaystyle\int_0^1 u'(x)w'(x)\,dx + \int_0^1 c(x)u(x)w(x)\,dx = w(1) \qquad \forall w \in \mathcal{V}, \end{array} \right. \tag{9}$$

where $\mathcal{V} = \{w \in H^1(0,1) \mid w(0) = 0\}$.

Let $u_c \in \mathcal{V}$ be the unique solution of the problem (9). We want to compute the sensitivity of the functional

$$\begin{array}{rccc} \mathcal{J}: & L^1(0,1) & \longrightarrow & \mathbf{R} \\ & c & \longmapsto & u_c(1). \end{array}$$

The associated Lagrangian is

$$\mathcal{L}(c,u,w) = u(1) + \int_0^1 u'(x)w'(x)\,dx + \int_0^1 c(x)u(x)w(x)\,dx - w(1).$$

The adjoint state is $p_c = -u_c$. It follows from lemma 2 that for all $\delta c \in L^1(0,1)$,

$$\mathcal{J}'(c).\delta c = -\int_0^1 \delta c(x) u_c^2(x)\,dx.$$

Let us detail this result with $c \equiv 0$ and δc_ϵ defined by:

$$\delta c_\epsilon = \left\{ \begin{array}{ll} 1 & \text{if } x \in [x_0, x_0 + \epsilon] \\ 0 & \text{elsewhere.} \end{array} \right.$$

The direct state u_0 is defined by $u_0(x) = x$ for all $x \in [0,1]$. According to theorem 2, the variation of the cost function j defined by $j(\epsilon) = \mathcal{J}(\delta c_\epsilon)$ is:

$$j(\epsilon) - j(0) = -\epsilon x_0^2 + o(\epsilon).$$

Let us check this result. When $c = \delta c_\epsilon$ the solution $u_\epsilon = u_{\delta c_\epsilon}$ can be explicitly computed:

$$u_\epsilon(x) = \begin{cases} \dfrac{2x}{(x_0+1)\exp(\epsilon) - (x_0-1)\exp(-\epsilon)} & \text{if } x \in [0, x_0] \\[2ex] \dfrac{(x_0+1)\exp(x-x_0) + (x_0-1)\exp(x_0-x)}{(x_0+1)\exp(\epsilon) - (x_0-1)\exp(-\epsilon)} & \text{if } x \in [x_0, x_0+\epsilon] \\[2ex] x - (x_0+\epsilon) + \dfrac{(x_0+1)\exp(\epsilon) + (x_0-1)\exp(-\epsilon)}{(x_0+1)\exp(\epsilon) - (x_0-1)\exp(-\epsilon)} & \text{if } x \in [x_0+\epsilon, 1] \end{cases}$$

Therefore

$$\begin{aligned} j(\epsilon) - j(0) &= u_\epsilon(1) - u_0(1) \\ &= -x_0^2\epsilon + x_0(x_0^2 - 1)\epsilon^2 + o(\epsilon^2) \end{aligned}$$

This confirms the result obtained using theorem 2.

2.4. Example 2

Let us consider the problem

$$\begin{cases} (cu')' = 0 & \text{in }]0,1[\\ u(0) = 0 \\ c(1)u'(1) = 1, \end{cases} \tag{10}$$

for $c \in L^\infty(0,1)$, and the cost function $\mathcal{J}(c) = u_c(1)$, where u_c is the solution to (10).

It is straightforward to check that the adjoint state is $p_c = -u_c$, and that

$$\mathcal{J}'(c)\delta c = -\int_0^1 \delta c(x)(u_c'(x))^2\,dx.$$

We consider $c \equiv c_0$ (a constant) and the perturbation defined by

$$\delta c_\epsilon = \begin{cases} \kappa & \text{if } x \in [x_0, x_0+\epsilon] \\ 0 & \text{elsewhere.} \end{cases}$$

After calculation of the exact solution u_ϵ, the value of the cost function is found to be

$$u_\epsilon(1) = \frac{1-\epsilon}{c_0} + \frac{\epsilon}{c_0+\kappa},$$

hence

$$j(\epsilon) - j(0) = \epsilon\left(\frac{1}{c_0+\kappa} - \frac{1}{c_0}\right). \tag{11}$$

On the other hand, the value of the gradient $\mathcal{J}'(c).\delta c$ is

$$-\epsilon(u_c')^2\kappa = -\epsilon\frac{\kappa}{c_0^2}. \tag{12}$$

Important remark: We observe that the classical gradient (12) is different from the topological gradient obtained in (11). In this example, theorem 1 can not be applied. The only possibility for the bilinear form a_c to be continuous is to have $c \in L^\infty$. The requirement in theorem 1 that $c \in L^p$ with $1 \leq p < 2$ is not met.

Note that if κ is small the expressions (11) and (12) are close, but when κ is close to $-c_0$ the classical gradient (12) remains bounded, while the topological gradient (11) goes to infinity .

3. First generalized adjoint method

In this section, we present a more general framework for topological sensitivity than theorem 2, since the conditions of application of this theorem appear to be restrictive in many interesting cases. In section 2.4, we noted that when the parameters belong to L^∞, the classical gradient is inadequate. We consider here the solution of a variational problem in a fixed Hilbert space, where both the bilinear and the linear form depend on a parameter. We give in theorem 3 the variation of a differentiable cost function. This section ends with an example and a counterexample.

3.1. The general frame

Let $\mathcal{V}$ be a fixed Hilbert space, and $\mathcal{L}(\mathcal{V})$ denote the space of linear continuous forms on $\mathcal{V}$ and $\mathcal{L}_2(\mathcal{V})$ the space of bilinear continuous forms on $\mathcal{V}$. For $\epsilon \geq 0$, let $a_\epsilon \in \mathcal{L}_2(\mathcal{V})$ and $\ell_\epsilon \in \mathcal{L}(\mathcal{V})$.

Hypothesis 2 *We make the following assumptions:*

2-a) *There exists a real function f, a bilinear form $\delta_a \in \mathcal{L}_2(\mathcal{V})$ and a linear form $\delta_\ell \in \mathcal{L}(\mathcal{V})$ such that*

$$f(\epsilon) \longrightarrow 0 \quad \text{when } \epsilon \overset{>}{\to} 0 \tag{13}$$

$$||a_\epsilon - a_0 - f(\epsilon)\delta_a||_{\mathcal{L}_2(\mathcal{V})} = o(f(\epsilon)) \tag{14}$$

$$||\ell_\epsilon - \ell_0 - f(\epsilon)\delta_\ell||_{\mathcal{L}(\mathcal{V})} = o(f(\epsilon)) \tag{15}$$

2-b) *the bilinear form a_0 is coercive : there exists a constant $\alpha > 0$ such that*

$$\forall u \in \mathcal{V}, \quad a_0(u,u) \geq \alpha ||u||^2.$$

According to (14), the bilinear form a_ϵ depends continuously on ϵ, hence there exist $\epsilon_0 > 0$ and $\beta > 0$ such that for every $\epsilon \in [0, \epsilon_0]$, the following uniform ellipticity condition holds:

$$\forall u \in \mathcal{V}, \quad a_\epsilon(u,u) \geq \beta ||u||^2.$$

According to Lax-Milgram's theorem, for $\epsilon \in [0, \epsilon_0]$ the following problem has a unique solution :

$$\begin{cases} \text{find } u_\epsilon \in \mathcal{V} \text{ such that} \\ a_\epsilon(u_\epsilon, v) = \ell_\epsilon(v), \qquad \forall v \in \mathcal{V}. \end{cases} \tag{16}$$

Lemma 3 *If hypothesis 2 holds, then*

$$||u_\epsilon - u_0||_{\mathcal{V}} = O(f(\epsilon)).$$

□

3.2. A generalized adjoint method

We consider now a cost function $j(\epsilon) = J(u_\epsilon)$, where the functional J is differentiable: for every $u \in \mathcal{V}$ there exists a linear continuous form $DJ(u) \in \mathcal{L}(\mathcal{V})$ such that

$$J(u+h) = J(u) + DJ(u).h + o(||h||_{\mathcal{V}}). \tag{17}$$

Under hypothesis 2-b), the adjoint problem:

$$\begin{cases} \text{find } p \in \mathcal{V} \text{ such that} \\ a_0(\psi, p) = -DJ(u_0)(\psi), \qquad \forall \psi \in \mathcal{V}. \end{cases} \tag{18}$$

admits a unique solution p_0 called the adjoint state.

For $\epsilon \geq 0$ we define the Lagrangian operator $\mathcal{L}_\epsilon$ by

$$\mathcal{L}_\epsilon(u,v) = J(u) + a_\epsilon(u,v) - \ell_\epsilon(v), \qquad \text{for } u, v \in \mathcal{V}.$$

The asymptotic expansion of $j : j(\epsilon) = J(u_\epsilon)$ is given by the following

Theorem 3 *If hypothesis 2 holds then*

$$j(\epsilon) - j(0) = f(\epsilon)\delta_{\mathcal{L}}(u_0, p_0) + o(f(\epsilon)),$$

where u_0 *is the solution of (16) for* $\epsilon = 0$, p_0 *is the solution of (18), and*

$$\forall u, v \in \mathcal{V}, \quad \delta_{\mathcal{L}}(u, v) = \delta_a(u, v) - \delta_\ell(v).$$

Proof: For all $v \in \mathcal{V}$ and $\epsilon \geq 0$,

$$j(\epsilon) = \mathcal{L}_\epsilon(u_\epsilon, v).$$

Setting $v = p_0$ we obtain

$$\begin{aligned} j(\epsilon) - j(0) =& \mathcal{L}_\epsilon(u_\epsilon, p_0) - \mathcal{L}_0(u_0, p_0) \\ =& J(u_\epsilon) - J(u_0) + a_\epsilon(u_\epsilon, p_0) - a_0(u_0, p_0) + \ell_0(p_0) - \ell_\epsilon(p_0) \\ =& J(u_\epsilon) - J(u_0) + [a_\epsilon(u_\epsilon, p_0) - a_0(u_\epsilon, p_0) + a_0(u_\epsilon - u_0, p_0)] \\ & - [\ell_\epsilon(p_0) - \ell_0(p_0) - f(\epsilon)\delta_\ell(p_0)] - f(\epsilon)\delta_\ell(p_0). \end{aligned}$$

Using (17), we know that

$$J(u_\epsilon) - J(u_0) = DJ(u_0).(u_\epsilon - u_0) + o(||u_\epsilon - u_0||_{\mathcal{V}}).$$

It follows that

$$\begin{aligned} j(\epsilon) - j(0) =& a_\epsilon(u_\epsilon, p_0) - a_0(u_\epsilon, p_0) + a_0(u_\epsilon - u_0, p_0) + DJ(u_0).(u_\epsilon - u_0) \\ & + o(||u_\epsilon - u_0||_{\mathcal{V}}) - [\ell_\epsilon(p_0) - \ell_0(p_0) - f(\epsilon)\delta_\ell(p_0)] - f(\epsilon)\delta_\ell(p_0). \end{aligned}$$

But p_0 is the adjoint state, hence

$$\begin{aligned} j(\epsilon) - j(0) =& a_\epsilon(u_\epsilon, p_0) - a_0(u_\epsilon, p_0) + o(||u_\epsilon - u_0||_{\mathcal{V}}) \\ & - [\ell_\epsilon(p_0) - \ell_0(p_0) - f(\epsilon)\delta_\ell(p_0)] - f(\epsilon)\delta_\ell(p_0). \\ =& (a_\epsilon - a_0)(u_0, p_0) + (a_\epsilon - a_0)(u_\epsilon - u_0, p_0) + o(||u_\epsilon - u_0||_{\mathcal{V}}) \\ & - [\ell_\epsilon(p_0) - \ell_0(p_0) - f(\epsilon)\delta_\ell(p_0)] - f(\epsilon)\delta_\ell(p_0). \end{aligned}$$

It follows from hypothesis 2-a) that

$$\begin{aligned} j(\epsilon) - j(0) =& f(\epsilon)\delta_a(u_0, p_0) + o(f(\epsilon)) + f(\epsilon)\delta_a(u_\epsilon - u_0, p_0) + o(f(\epsilon))||u_\epsilon - u_0||_{\mathcal{V}} \\ & + o(||u_\epsilon - u_0||_{\mathcal{V}}) - f(\epsilon)\delta_\ell(p_0). \end{aligned}$$

Finally, it follows from lemma 3, the continuity of δ_a and the fact that $f(\epsilon) \to 0$ when $\epsilon \to 0$ that

$$j(\epsilon) = j(0) + f(\epsilon)[\delta_a(u_0, p_0) - \delta_\ell(p_0)] + o(f(\epsilon)).$$

□

The generalized Lagrangian technique presented above have been applied to the several problems, we refer the reader to [GGM01, GuS01, Mas02, SAM03, HaM04, MPS05]. To satisfy the assumptions of hypothesis 2, we consider a domain truncation technique that we present in the next section on an example.

3.3. Example 3: Dirichlet condition for Poisson equation

We expose here the calculation of the topological sensitivity for a hole with a Dirichlet condition for the Poisson equation. We describe the domain truncation technique [GGM01, GuS01, Mas02, SAM03, HaM04, MPS05] that allows to work in a fixed Hilbert space and to satisfy Hypothesis 2.

Let $\Omega \subset \mathbf{R}^2$ with C^1 boundary. The boundary of Ω is decomposed in two parts $\partial\Omega = \Gamma_D \cup \Gamma_N$, with strictly positive measure. We consider a source term $h \in H_{00}^{1/2}(\Gamma_N)'$. Let $u_\Omega \in H^1(\Omega)$ be the unique solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \partial_n u = h & \text{on } \Gamma_N, \end{cases} \tag{19}$$

where $\partial_n u$ is the normal derivative of u. Let J be a differentiable function on $H^1(\Omega)$.

Our aim is to derive the topological sensitivity of J relatively to a circular perforation of radius ϵ with Dirichlet condition at some point $x \in \Omega$. The point $x \in \Omega$ is fixed, $\Omega_\epsilon = \Omega \setminus B(x, \epsilon)$ is the perforated domain (for ϵ small enough). The solution in the perforated domain is given by the problem

$$\begin{cases} \Delta u_{\Omega_\epsilon} = 0 & \text{in } \Omega_\epsilon \\ u_{\Omega_\epsilon} = 0 & \text{on } \Gamma_D \\ u_{\Omega_\epsilon} = 0 & \text{on } \Sigma_\epsilon \\ \partial_n u_{\Omega_\epsilon} = h & \text{on } \Gamma_N, \end{cases} \tag{20}$$

where Σ_ϵ is the boundary of $B(x, \epsilon)$.

Let $R > \epsilon$ be such that $B(x, R) \subset \Omega$. The boundary of $B(x, R)$ is denoted Σ_R and D_ϵ denotes the corona $B(x, R) \setminus B(x, \epsilon)$, see figure 1.

For a given $\psi \in H^{1/2}(\Sigma_R)$, we consider u_ψ^ϵ the (unique) solution to the problem

$$\begin{cases} \Delta u_\psi^\epsilon = 0 & \text{in } D_\epsilon \\ u_\psi^\epsilon = \psi & \text{on } \Sigma_R \\ u_\psi^\epsilon = 0 & \text{on } \Sigma_\epsilon, \end{cases}$$

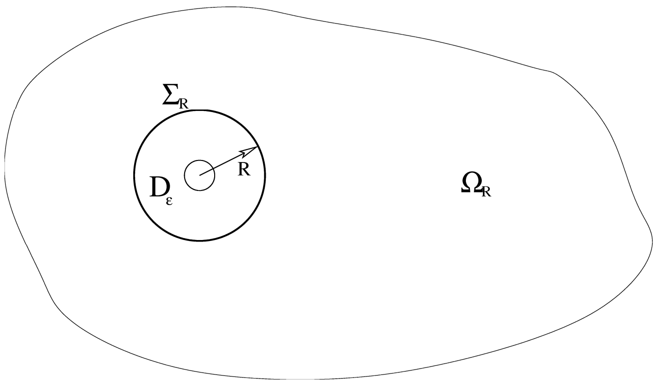

Figure 1. *Domain truncation.*

and the Dirichlet-to-Neumann operator

$$\begin{array}{cccc} T^\epsilon : & H^{1/2}(\Sigma_R) & \longrightarrow & H^{-1/2}(\Sigma_R) \\ & \psi & \longmapsto & T^\epsilon \psi = \nabla u^\epsilon_\psi . n|_{\Sigma_R}, \end{array}$$

where $n|_{\Sigma_R}$ is the outward normal to the boundary Σ_R.

We consider the truncated problem

$$\left\{ \begin{array}{ll} \Delta u_\epsilon = 0 & \text{in } \Omega_R \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \partial_n u_\epsilon + T^\epsilon u_\epsilon = 0 & \text{on } \Sigma_R \\ \partial_n u_\epsilon = h & \text{on } \Gamma_N, \end{array} \right. \tag{21}$$

the variational formulation associated to (21) is:

$$\left\{ \begin{array}{l} \text{find } u_\epsilon \in \mathcal{V}_R \text{ such that} \\ a_\epsilon(u_\epsilon, v) = \ell(v) \qquad \forall v \in \mathcal{V}_R, \end{array} \right.$$

with $\mathcal{V}_R = \{u \in H^1(\Omega_R) \mid u = 0 \text{ on } \Gamma_D\}$ and

$$a_\epsilon(u, v) = \int_{\Omega_R} \nabla u . \nabla v \, dx + \int_{\Sigma_R} (T^\epsilon u) v \, ds, \qquad \ell(v) = \int_{\Gamma_N} hv \, ds.$$

It is standard to prove that (21) has a unique solution in $\mathcal{V}_R$ that is the restriction to Ω_R of the solution of (20).

We now have a fixed Hilbert space, as required by theorem 3. It suffices to estimate the variation of the bilinear form a_ϵ (since in this example the linear form ℓ is independent of ϵ). We have [Mas02]:

$$a_\epsilon(u, v) - a_0(u, v) = \int_{\Sigma_R} ((T^\epsilon - T^0)u) v \, ds = f(\epsilon) \overline{u}^{\Sigma_R} \overline{p}^{\Sigma_R} + o(f(\epsilon)),$$

where $\overline{u}^{\Sigma_R}$, resp. $\overline{p}^{\Sigma_R}$, is the mean value of u, resp. p, on Σ_R. Since u_0 and p_0 are harmonic,

$$\delta a(u_0, p_0) = u_\Omega(x).p_\Omega(x),$$

where u_Ω is the solution and p_Ω is the adjoint state in Ω without a hole.

Using Theorem 3, we obtain the final result

$$j(\epsilon) - j(0) = f(\epsilon)u(x_0)p(x_0),$$

the topological gradient $g = u.p$ is exactly what we obtained by Theorem 2 using a straightforward way based on a penalization technique.

3.4. Example 4: interface between two material properties

Let $\mathcal{V} = \{u \in H^1(0,1) \mid u(0) = 0\}$. For $x_0 \in]0,1[$ and ϵ sufficiently small we define

$$a_\epsilon(u,v) = \int_0^{x_0+\epsilon} c_1 u'v'\,dx + \int_{x_0+\epsilon}^1 c_2 u'v'\,dx,$$

where c_1 and c_2 are two positive constants. We consider the following variational problem:

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ such that} \\ a_\epsilon(u,v) = \ell(v) \qquad \forall v \in \mathcal{V}, \end{cases} \tag{22}$$

where the linear form ℓ is given by: $\ell(v) = v(1)$.

We want to determine the variation of the cost function $j(\epsilon)$ defined by

$$j(\epsilon) = J(u_\epsilon) = u_\epsilon(1).$$

The solution of (22) is explicitly given by

$$u_\epsilon(x) = \begin{cases} \dfrac{1}{c_1}x & \text{for } x \in [0, x_0+\epsilon] \\ \\ \dfrac{1}{c_1}(x_0+\epsilon) + \dfrac{1}{c_2}(x - x_0 - \epsilon) & \text{for } x \geq x_0+\epsilon \end{cases}$$

It is straightforward to compute $j'(\epsilon)$ and check that $j'(0) = \dfrac{1}{c_1} - \dfrac{1}{c_2}$. On the other hand, let us compute the variation of the Lagrangian, that is δ_a (since $\delta_\ell = 0$). For $u, v \in \mathcal{V}$,

$$a_\epsilon(u,v) - a_0(u,v) = \int_{x_0}^{x_0+\epsilon} (c_1 - c_2)u'v'.$$

When $\epsilon \to 0$, this quantity is equivalent to $\epsilon(c_1 - c_2)u'(x_0^+)v'(x_0^+)$. Hence $a_\epsilon(u,v) - a_0(u,v) - \epsilon\delta_a(u,v) = o(\epsilon)$ with $\delta_a(u,v) = (c_1 - c_2)u'(x_0^+)v'(x_0^+)$.

Since $DJ(u_c).u = -\ell(u)$, the adjoint state is $p_0 = -u_0$, hence $\delta_{\mathcal{L}}(u_0, p_0) = \delta_a(u_0, p_0) = (c_1 - c_2)u_0'(x_0^+)v_0'(x_0^+) = \frac{c_2 - c_1}{c_2^2}$.

Important remark: We have $\delta_{\mathcal{L}}(u_0, p_0) \neq j'(\epsilon)$. In this example hypothesis 2-a) is not satisfied: the bilinear form δ_a is not continuous, since pointwise evaluation is not continuous on $L^2(0,1)$. Theorem 3 can not be applied. We suggest to the reader to apply theorem 3 when a domain truncation method is considered.

4. Second generalization of the adjoint approach

As shown in the previous example, in some cases of interest hypothesis 2 does not hold: the variation of the Lagrange operator with respect to the state is not small. An adaptation of theorem 3 is required in order to calculate topological sensitivity. The variation of the cost function is splitted in several terms, namely when the Lagrange operator is defined by

$$\mathcal{L}(\epsilon, u, v) = J(u) + a_\epsilon(u, v) - \ell_\epsilon(v),$$

the variation of cost function is given for any $v \in \mathcal{V}$ by

$$j(\epsilon) - j(0) = \mathcal{L}(\epsilon, u_\epsilon, v) - \mathcal{L}(0, u_0, v).$$

The variation of j must take into account $\partial\mathcal{L}/\partial\epsilon$, but also the variation induced by $u_\epsilon - u_0$. The variation of $u_\epsilon - u_0$ has been studied in [AVV01, AVV03, AmK04, MNP00].

See [Ams05] for the nonlinear case.

4.1. A general framework

We propose here a general result [ADS04], taking into account the variation of the Lagrange operator via the bilinear form a_ϵ, the linear form ℓ_ϵ, the state u_ϵ and also the cost function J_ϵ, in case it depends on ϵ.

Let $\mathcal{V}$ be a Hilbert space and, for $\epsilon \geq 0$, let a_ϵ be a bilinear continuous and coercive form on $\mathcal{V}$ and ℓ_ϵ be a linear continuous form on $\mathcal{V}$. Let u_ϵ be the direct state, solution to

$$\begin{cases} \text{find } u_\epsilon \in \mathcal{V} \text{ such that} \\ a_\epsilon(u_\epsilon, v) = \ell_\epsilon(v), \qquad \forall v \in \mathcal{V}. \end{cases}$$

We consider also a cost function $j(\epsilon) = J_\epsilon(u_\epsilon)$, where the functional J_ϵ is differentiable at the point u_0: there exists $L_\epsilon \in \mathcal{L}(\mathcal{V}, \mathbf{R})$ such that

$$J_\epsilon(u_0 + h) = J_\epsilon(u_0) + L_\epsilon(h) + o(||h||_{\mathcal{V}}).$$

Let p_ϵ be the adjoint state, solution to

$$\left\{\begin{array}{l}\text{find } p_\epsilon \in \mathcal{V} \text{ such that}\\ a_\epsilon(\psi, p_\epsilon) = -L_\epsilon(u_0).\psi, \qquad \forall \psi \in \mathcal{V}.\end{array}\right.$$

Hypothesis 3 *There exists a real function f tending to zero with ϵ, and four real numbers $\delta_a, \delta_\ell, \delta J_1, \delta J_2 \in \mathbf{R}$ such that*

3-a) $(a_\epsilon - a_0)(u_0, p_\epsilon) = f(\epsilon)\delta_a + o(f(\epsilon))$,

3-b) $(\ell_\epsilon - \ell_0)(p_\epsilon) = f(\epsilon)\delta_\ell + o(f(\epsilon))$,

3-c) $J_\epsilon(u_\epsilon) = J_\epsilon(u_0) + L_\epsilon(u_\epsilon - u_0) + f(\epsilon)\delta J_1 + o(f(\epsilon))$,

3-d) $J_\epsilon(u_0) = J_0(u_0) + f(\epsilon)\delta J_2 + o(f(\epsilon))$.

The quantity δJ_1 takes into account the variation of $u_\epsilon - u_0$ when it is not $O(f(\epsilon))$, and the quantity δJ_2 takes into account the variation of J_ϵ with respect to ϵ.

The topological sensitivity analysis of j is then given by

Theorem 4 *If hypothesis 3 holds, then*

$$j(\epsilon) - j(0) = f(\epsilon)\delta j + o(f(\epsilon)),$$

where $\delta j = \delta J_1 + \delta J_2 + \delta a - \delta \ell$.

Proof: We have

$$j(\epsilon) - j(0) = [J_\epsilon(u_\epsilon) - J_0(u_0)] + [a_\epsilon(u_\epsilon, p_\epsilon) - a_0(u_0, p_\epsilon)] - [\ell_\epsilon(_\epsilon) - \ell_0(p_\epsilon)].$$

it follows from hypothesis 3 and the definition of adjoint state that

$$\begin{aligned} j(\epsilon) - j(0) &= J_\epsilon(u_\epsilon) - J_0(u_0) + a_\epsilon(u_\epsilon - u_0, p_\epsilon) + f(\epsilon)(\delta a - \delta_\ell) + o(f(\epsilon)) \\ &= [J_\epsilon(u_\epsilon) - J_\epsilon(u_0)] + [J_\epsilon(u_0) - J_0(u_0)] + a_\epsilon(u_\epsilon - u_0, p_\epsilon) \\ &\quad + f(\epsilon)(\delta a - \delta_\ell) + o(f(\epsilon)) \\ &= a_\epsilon(u_\epsilon - u_0, p_\epsilon) + L_\epsilon(u_\epsilon - u_0) + f(\epsilon)(\delta J_1 + \delta J_1 + \delta a - \delta_\ell) + o(f(\epsilon)) \\ &= f(\epsilon)(\delta J_1 + \delta J_1 + \delta a - \delta_\ell) + o(f(\epsilon)). \end{aligned}$$

□

The topological expansion (Hypothesis 3) of a, ℓ, and J requires the knowledge of the topological asymtotic expansion of u_ϵ and p_ϵ. This subject is covered by a huge litterature [KoV87, FrV89, MNP00, Ili92, AVV01, AVV03, AmK04]. In the next section we present an example of application of these methods.

4.2. Insertion of inhomogeneities for Poisson equation

We present in this section the results obtained in [Ams03, ADS04].

4.2.1. The problem

We consider Ω a smooth domain in $\mathbf{R}^N$, $N = 2$ or 3. The boundary of Ω is divided in two parts of positive measure $\partial\Omega = \Gamma_N \cup \Gamma_D$. We assume that Ω contains a small inhomogeneity around the origin $\omega_\epsilon = \epsilon B$, where $B \subset \mathbf{R}^N$ is a bounded smooth domain containing the origin. We consider the piecewise constant function:

$$\alpha_\epsilon(x) = \begin{cases} \alpha_0 & \text{in } \Omega \setminus \omega_\epsilon \\ \alpha_1 & \text{in } \omega_\epsilon, \end{cases} \tag{23}$$

where α_0 and α_1 are positive constants. Let $g \in L^2(\Gamma)$. For $\epsilon \geq 0$, we denote u_ϵ the solution of the following Poisson problem:

$$\begin{cases} \nabla \cdot (\alpha_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\ \alpha_0 \partial_n u_\epsilon = g & \text{on } \Gamma_N, \\ u_\epsilon = 0 & \text{on } \Gamma_D. \end{cases} \tag{24}$$

This study is general, since a translation reduces the case of an inhomogeneity located around the point x_0 to an inhomogeneity located around the origin.

The variational formulation associated to (24) reads

$$\begin{cases} \text{find } u_\epsilon \in \mathcal{V} \text{ such that} \\ a_\epsilon(u_\epsilon, v) = \ell(v) \quad \forall v \in \mathcal{V}, \end{cases} \tag{25}$$

where $\mathcal{V} = \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}$ and, for $u, v \in H^1(\Omega)$,

$$a_\epsilon(u,v) = \int_\Omega \alpha_\epsilon \nabla u . \nabla v \, dx \qquad\qquad \ell(v) = \alpha_0 \int_{\partial\Omega} gv \, ds.$$

Since α_0 and α_1 are positive, the bilinear form a_ϵ is continuous and coercive on $\mathcal{V}$, and ℓ is a continuous linear form. Hence problem (25) admits a unique solution.

4.2.2. Variation of the bilinear form

Let $\mathbf{\Phi}$ be the solution of

$$\begin{cases} \Delta\mathbf{\Phi} = 0 \text{ in } B \text{ and } \mathbf{R}^N \setminus \overline{B}, \\ \mathbf{\Phi} \text{ is continuous across } \partial B, \\ \frac{\alpha_0}{\alpha_1}(\partial_n \mathbf{\Phi})^+ - (\partial_n \mathbf{\Phi})^- = -\mathbf{n} \text{ on } \partial\text{B}, \\ \lim_{|y|\to\infty} |\mathbf{\Phi}(y)| = 0, \end{cases}$$

where $\mathbf{n}$ denotes the outward unit normal to ∂B and superscript $+$, resp. $-$, denotes the limits over ∂B from outside, resp. from inside.

It is proved in [ADS04] that hypothesis 3-a) is satisfied with $f(\epsilon) = \epsilon^N$ and

$$\delta a = (\alpha_1 - \alpha_0)\nabla u_0(0)^T \left[\left(\frac{\alpha_0}{\alpha_1} - 1\right)\int_{\partial B} \mathbf{n} \otimes \mathbf{\Phi}(y)\, ds(y) + |B|I\right] \nabla p_0(0),$$

where $u \otimes v = uv^T$. Let us introduce the matrix

$$P = \left(\frac{\alpha_0}{\alpha_1} - 1\right)\int_{\partial B} \mathbf{n} \otimes \mathbf{\Phi}(y)\, ds(y) \tag{26}$$

The matrix P depends only on the shape of B and of the ratio α_0/α_1, and we have

$$\delta a = (\alpha_1 - \alpha_0)\nabla u_0(0)^T (P + |B|I)\nabla v_0(0)^T.$$

Unit ball: when B is the unit ball, the matrix P defined in (26) is equal to

$$P = \frac{\alpha_0 - \alpha_1}{(N-1)\alpha_0 + \alpha_1}|B|I.$$

Ellipse in the plane: when B is the ellipse in the plane with major axis $2r_1$ and minor axis $2r_2$, the matrix P is equal to

$$P = \pi r_1 r_2 (\alpha_0 - \alpha_1)\begin{pmatrix} \dfrac{1}{\alpha_0 r_1 + \alpha_1 r_2} & 0 \\ 0 & \dfrac{1}{\alpha_0 r_2 + \alpha_1 r_1} \end{pmatrix}$$

Elliptic hole with homogeneous Neumann condition: it is the limiting case of the ellipse with $\alpha_1 \to 0$. We obtain formally a well known result [SoZ99]

$$P = \pi \begin{pmatrix} r_2 & 0 \\ 0 & r_1 \end{pmatrix}$$

and δa is straightforward to estimate.

Straight crack of length $2r_1$ with normal $\mathbf{n}$: it is formally the limit of the previous case with $r_2 \to 0$. After calculations, one obtains

$$\delta a = -\pi r_1 (\nabla u_0(0).\mathbf{n})(\nabla p_0(0).\mathbf{n}). \tag{27}$$

This formula is proved to be exact using other methods in [AHM05].

The linear form ℓ does not depend on ϵ, hence

$$\delta \ell = 0.$$

4.2.3. Variation of the cost function

The calculation of δJ_1 and δJ_2 is given in [ADS04] for different cost functions.

Example 1: the cost function is defined by

$$J_\epsilon(u) = \int_\Omega \alpha_\epsilon |u_\epsilon - u_g|^2 \, dx$$

where $u_g \in H^2(\Omega)$ is a "target" function.

Hypothesis 3 is satisfied with $f(\epsilon) = \epsilon^N$, $L_\epsilon(u) = 2\int_\Omega \alpha_\epsilon u(u_0 - u_g)\, dx$ for all $u \in H^1(\Omega)$, $\delta J_1 = 0$ and $\delta J_2 = (\alpha_1 - \alpha_0)|B|\,|u_0(0) - u_g(0)|^2$.

Example 2: The cost function is defined by

$$J_\epsilon(u) = \int_\Omega \alpha_\epsilon |\nabla(u_\epsilon - u_g)|^2 \, dx$$

where $u_g \in H^3(\Omega)$ is a "target" function.

Hypothesis 3 is satisfied with $f(\epsilon) = \epsilon^N$, $L_\epsilon(u) = 2\int_\Omega \alpha_\epsilon \nabla u \nabla(u_0 - u_g)\, dx$ for all $u \in H^1(\Omega)$, $\delta J_1 = (\alpha_1 - \alpha_0)\nabla u_0(0)^T P \nabla u_0(0)$ and
$\delta J_2 = (\alpha_1 - \alpha_0)|B|\,|\nabla u_0(0) - \nabla u_g(0)|^2$, where the matrix P is defined by (26).

4.3. Crack detection for Poisson equation: numerical results

We present here results obtained in [AHM05]. A domain contains an unknown crack, and the available data are measures of u and $\partial_n u$ on the boundary of the domain, where u satisfies a Poisson equation in the domain and the crack is perfectly insulating.

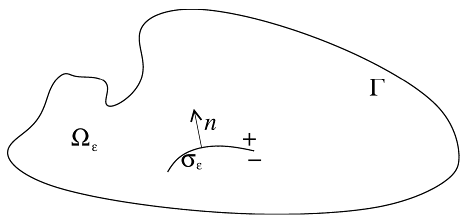

Figure 2. *The cracked domain.*

4.3.1. The problem

Let Ω be a domain containing a perfectly insulating crack σ^* whose location, orientation, shape and length are to be retrieved. We dispose of the temperature θ measured on the boundary Γ for a heat flux φ prescribed: $\theta = u(\sigma^*)_{|\Gamma}$, where $u(\sigma^*)$ is the solution to

$$\begin{cases} \Delta u(\sigma^*) &= 0 & \text{in} & \Omega \setminus \overline{\sigma^*}, \\ \partial_n u(\sigma^*) &= \varphi & \text{on} & \Gamma, \\ \partial_n u(\sigma^*) &= 0 & \text{on} & \sigma^*. \end{cases} \tag{28}$$

To ensure well-posedness of the above system, we assume the normalization condition

$$\int_\Gamma \varphi ds = 0$$

and we impose that the mean value of the solution is equal to zero:

$$\int_{\Omega\setminus\overline{\sigma^*}} u(\sigma^*)dx = 0.$$

Since the boundary conditions (θ, φ) are over specified, we can define for any crack $\sigma \subset \Omega$ two direct problems similar to (28): the "Dirichlet" problem with Dirichlet boundary condition $u = \theta$ on Γ, and the "Neumann" problem with Neumann boundary condition $\partial_n u = \varphi$ on Γ. The solutions of these direct problems are denoted u_D and u_N. The actual crack σ^* is reached ($\sigma = \sigma^*$) when there is no misfit between both solutions, that is, when the cost functional

$$\mathcal{J}(\sigma) = J(u_D(\sigma), u_N(\sigma)) = \frac{1}{2}\|u_D(\sigma) - u_N(\sigma)\|^2_{L^2(\Omega)} \tag{29}$$

vanishes. This is the so-called Kohn-Vogelius criterion [KoV87]. To compute the corresponding topological gradient, we need to solve numerically:

– two direct problems in the full domain Ω, one Dirichlet problem and one Neumann problem. Their solutions are denoted by u_D and u_N.

– two adjoint problems (defined also in the full domain Ω) that are derived directly from their variational formulations. Their solutions are denoted by p_D and p_N.

The following value of the topological sensitivity is a straightforward application of (27), a different proof can be found in [AHM05]:

$$\mathcal{J}(\sigma_{x,\epsilon,\mathbf{n}}) - \mathcal{J}(\emptyset) = -\pi\epsilon^2 \left[(\nabla u_D(x).\mathbf{n})(\nabla p_D(x).\mathbf{n}) + (\nabla u_N(x).\mathbf{n})(\nabla p_N(x).\mathbf{n})\right] + o(\epsilon^2),$$

where $\sigma_{x,\epsilon,\mathbf{n}}$ is the line crack of length 2ϵ, centered at the point x and of unit normal $\mathbf{n}$. The corresponding topological gradient

$$g(x, \mathbf{n}) = -\pi[(\nabla u_D(x).\mathbf{n})(\nabla p_D(x).\mathbf{n}) + (\nabla u_N(x).\mathbf{n})(\nabla p_N(x).\mathbf{n})]$$

can be written as follows:

$$g(x, \mathbf{n}) = \mathbf{n}^T M(x)\mathbf{n}, \tag{30}$$

where $M(x)$ is the symmetric matrix defined by

$$M(x) = -\frac{\pi}{2}\left[\nabla u_D(x)\nabla p_D(x)^T + \nabla p_D(x)\nabla u_D(x)^T + \nabla u_N(x)\nabla p_N(x)^T \right.$$
$$\left. + \nabla p_N(x)\nabla u_N(x)^T\right].$$

According to that expression, $g(x, \mathbf{n})$ is minimal at the point x when the normal $\mathbf{n} = \mathbf{n_1}$ is an eigenvector associated to the smallest eigenvalue $\lambda_1(x)$ of the matrix $M(x)$. Then, $g(x, \mathbf{n_1}) = \lambda_1(x)$. We will call topological gradient this value.

4.3.2. Numerical results in one iteration without noise

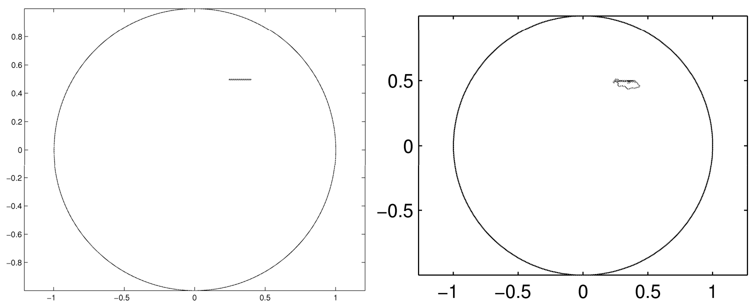

Figure 3. *Left : the unknown crack ; right : superposition of the actual crack and a negative isovalue of the topological gradient (courtesy S.Amstutz, I.Horchani).*

This leads to a simple and very fast numerical procedure. First, the two direct problems and the two adjoint problems (Dirichlet and Neumann) are solved. Then the matrix $M(x)$ and its eigenvalues are computed in each cell of the mesh. By using the previous asymptotic analysis, the crack is expected to lie in the regions where the topological gradient is the most negative.

The heat flux φ is imposed on Γ by $\varphi(x) = x_2$, the second coordinate of the point x. In this experiment, the flux inside the full domain is not parallel to the crack, so only one measurement is needed for the reconstruction (see [AnB96]). We apply formula (30), the location of the unknown crack and the topological gradient are indicated in figure 3. We observe that the most negative values of the topological gradient are located near the actual crack.

4.3.3. Numerical results in one iteration with noise

We focus here on simulated noisy measurements. A white noise is added to the exact data. Figure 4 shows the results obtained for 5%, and 20% of noise. We observe that the inversion procedure is quite robust with respect to the presence of noise in the measurements.

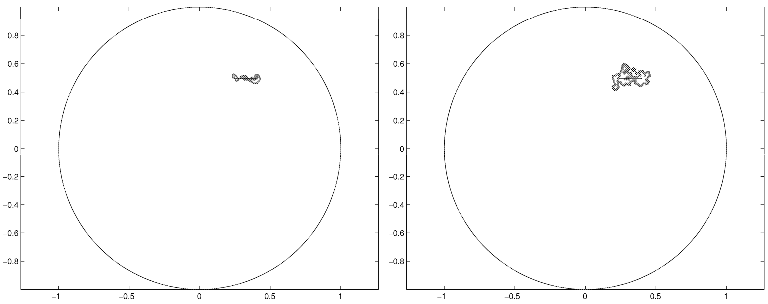

Figure 4. *Negative isovalues of the topological gradient. Noise: 5% (left) and 20% (right). (courtesy S.Amstutz, I.Horchani).*

4.3.4. Case of multi-cracks

The computation of the topological gradient does not depend on the number of cracks inside the domain. This remark is illustrated by the following experiment. The actual cracks and the topological gradient are represented in Figure 5. We use now two fluxes $\varphi_1(x) = x_1$ and $\varphi_2(x) = x_2$. The cost functional is the sum of the two quadratic misfits. We emphasize again that these results are obtained in only one iteration.

5. References

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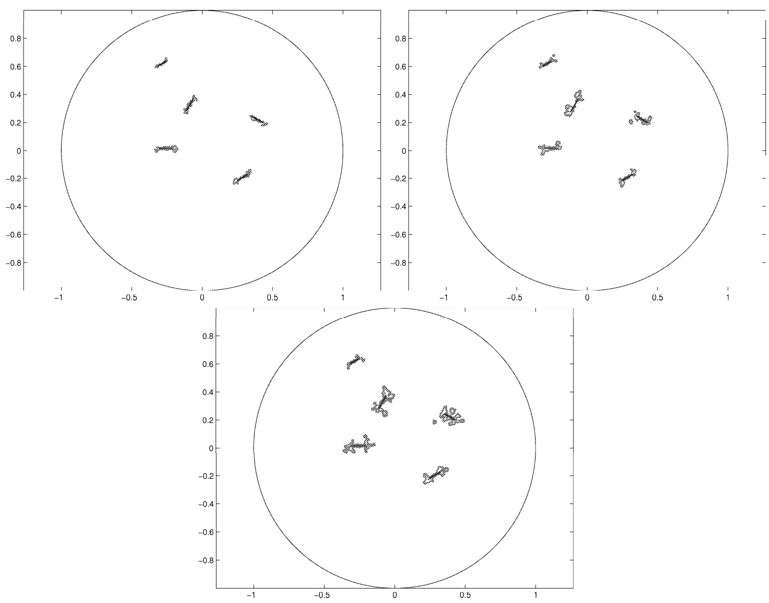

Figure 5. *Isovalues of the topological gradient. Noise: 5% (left top), 10% (right top), and 20% (down) (courtesy S.Amstutz, I.Horchani).*

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