

An Algorithm for the Navier-Stokes Problem

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RÉSUMÉ. Cette étude est la continuation des travaux [7],[8] et [9] qui sont basés sur l'étude faite par Glowinski & Al [3] et [4] ainsi que Bernardi & Al (voir [1] et [2]). Ici nous proposons un Algorithme pour résoudre un problème non-linéaire issu de la mécanique des fluides. Dans [7] nous avons étudié le problème de Stokes en adaptant la technique de Glowinski, grâce à laquelle, on peut découpler la pression de la vitesse lors de la résolution du problème de Stokes. Dans ce travail, nous étendons notre étude et montrons que cette technique peut être utilisée dans la résolution d'un problème non-linéaire comme les équations de Navier Stokes. Des tests numériques confirment l'intérêt de la discrétisation.

ABSTRACT. This study is a continuation of the one done in [7],[8] and [9] which are based on the work, first derived by Glowinski & Al in [3] and [4] and also Bernardi & Al [1] and [2]. Here, we propose an Algorithm to solve a nonlinear problem arising from fluid mechanics. In [7], we have studied Stokes problem by adapting Glowinski technique. This technique is useful as it decouples the pressure from the velocity during the resolution of the Stokes problem. In this paper, we extend our study to show that this technique can be used in solving a nonlinear problem such as the Navier Stokes equations. Numerical experiments confirm the interest of this discretisation.

MOTS-CLÉS : Algorithmes Numériques, Approximations Spectrales, Problèmes de Stokes & Navier-Stokes.

KEYWORDS : Numerical Algorithms, Spectral Approximations, Stokes & Navier Stokes Problems.
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1. Introduction

Over the last decade, there has been considerable interest in studying axisymmetric problems and their approximations. These problems have been studied by many authors, namely, Glowinski & Al [3],[4], Bernardi & Al [1],[2] and Nouri & Al [7],[8],[9] and references therein. The numerical discretisations relying on the current, the whirlpool and the pressure formulation has first been performed for finite element methods see [10], in the much simpler case of the Stokes problem, and extended to the case of spectral methods [2] and [7], where spectral analogues of Nédelec finite elements [10] are used. Relying on the work in [7] and [8], we propose a discretisation of the Navier Stokes equation in the situation where the domain is axi-symmetric. We present some numerical experiments which confirm the optimality of the discretisation. The analysis of Navier-Stokes equations for incompressible viscous flows in a connected bounded domain can be reduced to analysing the corresponding Stokes problem.

First, we consider the Stokes problem in the tridimensional domain $\tilde{\Omega}$ that we suppose invariant by rotation around the axis of the cylinder. Then, we use the cylindrical coordinates (r, θ, z) , $r \in IR^+$, $\theta \in [-\pi, \pi]$ and $z \in IR$. The domain $\tilde{\Omega}$ is denoted by :

$$\tilde{\Omega} = \{(x, y, z) \in IR^3 / (r, z) \in \Omega \cup \Gamma_0 \text{ and } -\pi \leq \theta \leq \pi\},$$

where Γ_0 is the interior part of the boundary of Ω in the axis $\{r = 0\}$ and we denote by $\Gamma = \partial\Omega / \Gamma_0$.

First, we introduce the functional spaces :

$$L_1^2(\Omega) = \{v \text{ integrable in } \Omega, \text{ such that } \int_{\Omega} v^2(r, z) r dr dz < \infty\}$$

with the norm

$$\|v\|_{L_1^2(\Omega)} = \left(\int_{\Omega} v^2(r, z) r dr dz < \infty \right)^{\frac{1}{2}}, \quad \forall v \in L_1^2(\Omega).$$

and the corresponding weighted Sobolev spaces :

$$H_1^1(\Omega) = \{v \in L_1^2(\Omega), \quad \nabla v = (\partial_r v, \partial_z v) \in (L_1^2(\Omega))^2\}.$$

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$$H_{1,0}^1(\Omega) = \{v \in H_1^1(\Omega) : v = 0 \text{ on } \partial\Omega/\Gamma_0\}.$$

$$V_1^s(\Omega) = \{v \in H_1^s(\Omega), \quad r^{l+m-s} \partial_r^l \partial_z^m v \in L_1^2(\Omega), \quad 0 \leq l + m \leq s\}$$

$$V_{1,0}^1(\Omega) = \{v \in V_1^1(\Omega), \quad v = 0 \text{ on } \Gamma\}.$$

Then, the Stokes problem can be written as :

$$\begin{cases} -\nu(\partial_r^2 u_r + \frac{1}{r} \partial_r u_r - \frac{1}{r^2} u_r + \partial_r^2 u_z + \partial_z^2 u_z) + \partial_r p = f_r, & \text{in } \Omega \\ -\nu(\partial_r^2 u_z + \frac{1}{r} \partial_r u_z + \partial_z^2 u_z) + \partial_z p = f_z, & \text{in } \Omega \\ \partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0, & \text{in } \Omega \\ u_r = 0, u_z = 0 & \text{on } \Gamma. \end{cases} \quad [1]$$

For the known $f = (f_r, f_z) \in (V_0^1)' \times (H_0^1)'$, there exists $u = (u_r, u_z) \in (V_{1,0}^1) \times (H_{1,0}^1)$. The equation for the divergence can be written as :

$$\partial_r(ru_r) + \partial_z(ru_z) = 0, \quad \text{in } \Omega \quad [2]$$

Equation [2] show that the vector (ru_r, ru_z) is of divergence free. From theorem 3.1 in chapter 1 of [5], there exists a function φ verifying

$$\begin{pmatrix} ru_r \\ ru_z \end{pmatrix} = ru = \text{rot}(\varphi) = \begin{pmatrix} \partial_z \varphi \\ -\partial_r \varphi \end{pmatrix}.$$

By taking $\varphi = r\psi$, we have $\partial_z \varphi = r\partial_z \psi$ and $-\partial_r \varphi = -\psi - r\partial_r \psi$, then we set :

$$\begin{cases} u_r = \partial_z \psi \\ u_z = -\frac{1}{r}(\partial_r(r\psi)) \end{cases} \quad [3]$$

Now, let us denote [2] by $\text{div}_r u$ which can also be written as

$$\partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0 \quad [4]$$

and by $\text{rot}_r \psi$ the vector $(\partial_z \psi, -\frac{1}{r} \partial_r(r\psi))$. We also introduce the operator Rot given by :

$$\text{Rot}(v) = \partial_r v_z - \partial_z v_r, \quad v = (v_r, v_z) \quad [5]$$

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and the operator rot such that for any ψ , we have :

$$rot\psi = \begin{pmatrix} \partial_z\psi \\ -\partial_r\psi \end{pmatrix} \quad [6]$$

We can then write

$$Rot(rot\psi) = -\Delta\psi \quad [7]$$

Remark 1

If $divv = 0$, we have $rotRot(v) = -\Delta v$. If we define the operator Δ_r by :

$$\Delta_r\psi = \partial_r^2\psi + \frac{1}{r}\partial_r\psi - \frac{1}{r^2}\psi + \partial_z^2\psi \quad [8]$$

we can then show that

$$Rot(rot_r\psi) = -\Delta_r\psi \quad [9]$$

(For details see proposition 3.3.2 in [7]).

2. Continuous Problem

The question is now what are the necessary boundary conditions to ensure the existence and uniqueness of the current function ψ defined by :

$$\begin{cases} u_r = \partial_z\psi \\ u_z = -\frac{1}{r}\partial_r(r\psi) \end{cases} \quad [10]$$

and then deduce its regularity from the regularity of the velocity u , the solution of the Stokes problem, see [7] for details. We consider

$$H_1(div_r, \Omega) = \{v \in (L_1^2(\Omega))^2 / div_r v \in L_1^2(\Omega)\}$$

with the norm :

$$\|v\|_{H_1(div_r, \Omega)} = \left(\|v\|_{(L_1^2(\Omega))^2}^2 + \|div_r v\|_{L_1^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad [11]$$

Note that the boundary operator is defined and continuous from $H_1^1(\Omega)$ onto $H_1^{\frac{1}{2}}(\Gamma)$.

Note also that $H_1^{-\frac{1}{2}}(\Gamma)$ coincides with $H_1^{\frac{1}{2}}(\Gamma)$ in the points away from $\overline{\Gamma_0} \cap \overline{\Gamma}$. In the same

way, we define the normal boundary operator which is continuous from $H_1^1(div_r, \Omega)$ onto $H_1^{-\frac{1}{2}}(\Gamma)$ (The dual of $H_1^{\frac{1}{2}}(\Gamma)$) by :

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$$\forall v \in H_1(\operatorname{div}_r, \Omega) \text{ and } q \in H_1^1(\Omega),$$

$$(v.n, q)_\Gamma = \int_{\Omega} (\operatorname{div}_r v . q)(r, z) r dr dz + \int_{\Omega} (v . \nabla q)(r, z) r dr dz. \quad [12]$$

$(., .)_\Gamma$ is the scalar product of duality between $H_1^{\frac{1}{2}}(\Gamma)$ and $H_1^{-\frac{1}{2}}(\Gamma)$ with the measure $d\tau = r dr$ on the side which is parallel to oz and $= dz$ on the two sides of Γ .

It has been shown in [7] that :

Proposition 1

$\forall v \in H_1(\operatorname{div}_r, \Omega)$ such that $\operatorname{div}_r v = 0$ and $v.n|_\Gamma = 0$, $\exists \varphi \in V_1^1(\Omega)$ such that $v = \operatorname{rot}_r \varphi$.

From theorem 3.1 of [5], $\exists \varphi \in H^1(\Omega)$ such that $rv = \operatorname{rot} \varphi$. If we set $\Phi = \frac{1}{r} \varphi$ verifying $v = \operatorname{rot}_r \Phi$, we have $\partial_z \Phi \in L_1^2(\Omega)$ and $\frac{1}{r} \partial_r(r\Phi) \in L_1^2(\Omega)$. By integrating by parts, we get :

$$\frac{\partial(r\Phi)}{\partial\tau} = 0 \text{ on } \Gamma. \quad [13]$$

where $\frac{\partial\varphi}{\partial\tau}$ is the tangential derivative given by :

$$\frac{\partial\varphi}{\partial\tau} = \nabla\varphi . \tau. \quad [14]$$

The function $r\Phi = C_i$ on each connected component $\Gamma_i|_{i=1}^I$ of $\partial\Omega$ where $C_i = 0$ on all the ones that touch the axis $\{r = 0\}$. Hence $\varphi \in V_1^1(\Omega)$.

Let us now define the boundary conditions for the current function. For, we consider $v \in H_1(\operatorname{div}_r, \Omega)$ such that $\operatorname{div}_r v = 0$. Let us denote by φ the function associated to v , From proposition 1, we have $r\varphi = c$ (a constant) on Γ . However Γ touch the axis $\{r = 0\}$, hence $c = 0$ and $\varphi|_\Gamma = 0$. We set the following result :

Corollary 2

Let $v \in H_1(\operatorname{div}_r, \Omega)$ such that $\operatorname{div}_r v = 0$ and $v.n|_\Gamma = 0$. If φ is the current function associated to v , then φ is the solution of the following problem :

$$\begin{cases} \varphi \in V_{1,0}^1(\Omega) \\ \int (\operatorname{rot}_r \varphi . \operatorname{rot}_r \chi)(r, z) r dr dz = \int (v . \operatorname{rot}_r \chi)(r, z) r dr dz, \forall \chi \in V_{1,0}^1(\Omega) \end{cases} \quad [15]$$

To study this problem (see [5] and [7]), we introduced the space :

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$$H_1(rot, \Omega) = \{v \in (L_1^2(\Omega))^2 / rot(v) \in L_1^2(\Omega)\} \tag{16}$$

and we define the boundary operator from $H_1(rot, \Omega)$ onto $H^{-\frac{1}{2}}(\Gamma)$ as :

$$\forall v \in H_1(rot, \Omega), \forall \chi \in V_1^1(\Omega),$$

$$\langle v.\tau, \chi \rangle_\Gamma = \int_\Omega (v.rot_r \chi)(r, z) r dr dz - \int_\Omega rot(v)\chi(r, z) r dr dz \tag{17}$$

Hence we can write :

$$\begin{cases} -\Delta_r \varphi = rot(v) \text{ in } D'(\Omega) \\ \varphi|_\Gamma = 0 \\ (\frac{1}{r} \frac{\partial(r\varphi)}{\partial n} + v.\tau)|_\Gamma = 0 \end{cases} \tag{18}$$

where $\frac{\partial \varphi}{\partial n} = \nabla \varphi . n$.

Proposition 3

We suppose that $v \in V_{1,0}^1(\Omega) \times H_{1,0}^1(\Omega) / div_r(v) = 0$ and $v.n|_\Gamma = 0$, then there exists a current function φ in $V_1^2(\Omega) / v = rot \varphi$ verifying $\varphi = 0$ on $\partial\Omega$ and $\frac{\partial \varphi}{\partial n} = 0$ on Γ , where $V_1^2(\Omega)$ is defined by :

$$V_1^2(\Omega) = \{ \varphi \in L_1^2(\Omega) / r^{l+m-2} \partial_r^l \partial_z^m \varphi \in L_1^2(\Omega), 0 \leq (l+m) \leq 2 \} .$$

Remark 2

For the proof of this proposition, we use Hardy inequality (See [7] and references therein).

Remark 3

In [8], we showed how we can generalise this result.

3. Current-Whirpool function formulation

In this section, we introduce a new unknown called whirlpool function $\omega = rot(u)$ which gives :

$$-\Delta u = rot_r \omega \tag{19}$$

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Δ is applied to (u_r, u_z) in the two first equations of [1].

If we substitute this result in [1], we obtain

$$\begin{cases} -v\partial_z\omega + \partial_r p = f_r \\ -v\frac{1}{r}\partial_r(r\omega) + \partial_z p = f_z \end{cases} \quad [20]$$

The interesting point of this formulation is to decouple the pressure from the velocity to obtain a separate problem for the pressure. We have

$$-\Delta_r\omega = \frac{1}{v}(\partial_z f_r - \partial_r f_z) \quad [21]$$

where $u = rot_r\psi$ and $\omega = -\Delta_r\psi$.

Hence we can conclude that this formulation is equivalent to the one derived by Glowinski and Pironneau in [3] and [4].

3.1. The reduced Problem

The advantage of this decoupling is to write [1] as :

$$(P) \begin{cases} -\Delta_r\omega = rot(f) & \text{on } \Omega \\ -\Delta_r\psi = \omega & \text{on } \Omega \\ \psi|_{\Gamma} = 0 \\ \frac{\partial\psi}{\partial n}|_{\Gamma} = 0 \end{cases}$$

Problem (P) is exactly a Dirichlet problem for the biharmonic operator Δ_r^2 , where the solution is the current function ψ such that :

$$(P_0) \begin{cases} -\Delta_r^2\psi = rot(f) & \text{on } \Omega \\ \psi|_{\Gamma} = 0 \\ \frac{\partial\psi}{\partial n}|_{\Gamma} = 0 \end{cases}$$

Problem (P_0) can be written in a matrix form as :

$$DU = F$$

where D is the matrix with the entries $a_N(h_i l_j, h_r l_s)$ defined by :

$$a_N(h_i l_j, h_r l_s) = (h'_i l'_j, h'_r l'_s)_N + (h_i l'_j, h_r l'_s)_N + \left(\frac{1}{r}h_i l_j, \frac{1}{r}h_r l_s\right)_N,$$

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where $(u, v)_N$ is given by :

$$(u, v)_N = \sum_{i=1}^{N+1} \sum_{j=0}^N u(\zeta_i, \xi_j) v(\zeta_i, \xi_j) \omega_i \rho_j, \quad 2 \leq i, r \leq N, \quad 1 \leq j, s \leq N-1.$$

Here h_i respectively l_j , are the Lagrange polynomials associated at the nodes ζ_i and ξ_j , $1 \leq i \leq N+1$ and $0 \leq j \leq N$.

U is constructed by the values of the solution u_N at the nodes (ζ_i, ξ_j) and F by the terms, $f(\zeta_i, \xi_j) \omega_i \rho_j$, where ω_i and ρ_j are the associated weights.

We remark that D is a symmetric positive definite matrix, therefore the gradient conjugate Algorithm can be used. A study of the continuous problem was done by [2] and [3], and the spectral discretisation was studied in [7] and [9]. Here we would like to extend this study to the Navier-Stokes equation.

4. Numerical Study

In the same way we have applied the above analysis to a nonlinear problem (see [9] for details) and analogous results have been found. Using the above formulation, we can write the Navier Stokes equation as :

$$(P) \begin{cases} -\nu \Delta_r \omega + \nabla_r \psi \cdot \text{rot}_r \omega = \text{rot}(f) & \text{on } \Omega \\ -\Delta_r \psi = \omega & \text{on } \Omega \\ \psi|_{\Gamma} = 0 \\ \frac{\partial \psi}{\partial \eta}|_{\Gamma} = 0 \end{cases}$$

where

$$\nabla_r \varphi = \left(\frac{1}{r} \partial_r(r\varphi), \partial_z \varphi \right)$$

and

$$\text{rot}_r \varphi = \left(\partial_z \varphi, -\frac{1}{r} \partial_r(r\varphi) \right) \quad \text{for any regular function } \varphi.$$

Therefore we set an Algorithm to solve problem (P).

4.1. Algorithm

We propose the following Algorithm

- Approximate ψ and $\{\psi_m\}_{m=1}^M$ by :

$$\psi_{m+1} = \psi_m + \alpha(\bar{\psi}_m - \psi_m).$$

- Approximate ω and $\{\omega_m\}_{m=1}^M$ by :

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$$\omega_{m+1} = \omega_m + \alpha(\bar{\omega}_m - \omega_m).$$

ω_m and ψ_m are solutions of

$$\begin{cases} -\nu \Delta_r \bar{\omega}_m = -\frac{1}{r} \partial_r (r \psi_m) \partial_z \omega_m - \partial_r (r \omega_m) \partial_z \psi_m + \text{rot}(f) \\ -\Delta_r \bar{\psi}_m = \bar{\omega}_m \\ \bar{\psi}_m|_{\Gamma} = 0 \quad \text{and} \quad \frac{\partial \bar{\omega}_m}{\partial \eta}|_{\Gamma} = 0 \end{cases}$$

Hence the nonlinear problem can be solved as four problems for Δ_r .

For the numerical tests, we have taken the following examples.

Example 1

The first test is the function

$$\Psi = \cos(\pi r) \sin(\pi z) (1 - r^2)^3 (1 - z^2)$$

Example 2

The second test is a singular stream

$$\Psi = (1 - r)^3 (1 + r)^{11/2} (1 - z^2)$$

Numerical tests were carried out in the IRISA laboratory in Rennes (France)

4.2. Numerical Results

Numerical experiments turn out to be in agreement with the analysis done in [7] and [9]. As shown in Figures 1, 2 and 3, numerical results for examples 1 and 2 were compared to the exact solutions. The error plots are in the L^2 -norm. We noticed that :

- 1- The error analysis for the current function is qualitatively the same for both examples 1 and 2 (Figure 3).
- 2- The choice of the basis does not affect the quality of our numerical results. However, when using the nodes to be the roots or the extremums of Chebyshev polynomials, better results were found.
- 3- From Figure 2, where numerical investigations for a singular problem (example 2) was considered, the error has a spectral behavior, that is to say, we do not have any spurious modes that destruct the solution.
- 4- Many choices for the relaxation parameter were used and the optimal value was $\alpha = 0.5$. For the choice of parameters for our plots, we have taken the viscosity $\nu = 1$, the

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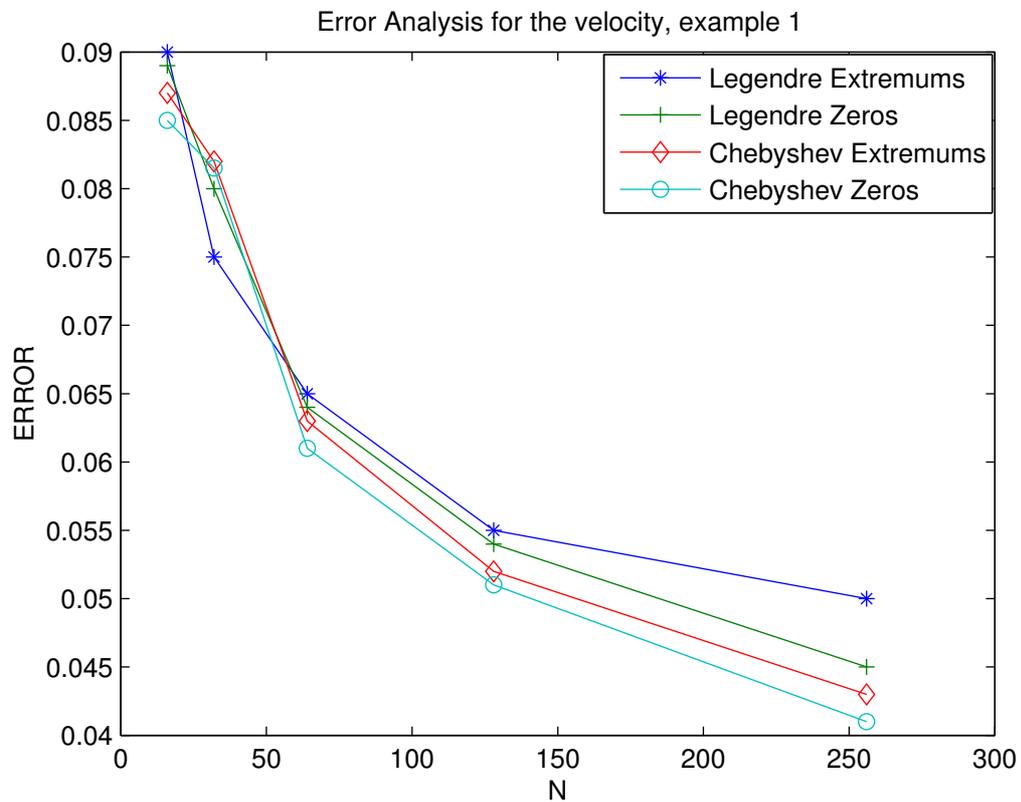


Figure 1. Error Analysis for the velocity, Example 1

relaxation parameter $\alpha = 0.5$.

A Study of the pressure is under consideration.

5. Conclusion

In this paper, we proposed a new technique similar to the one by Glowinski & Al [5] and show that we can reduce Stokes and Navier-Stokes problems to a set of saddle point ones ; where the unknowns are the current and whirlpool functions. Here the pressure is completely decoupled from the velocity to give a simple Dirichlet problem to be solved.

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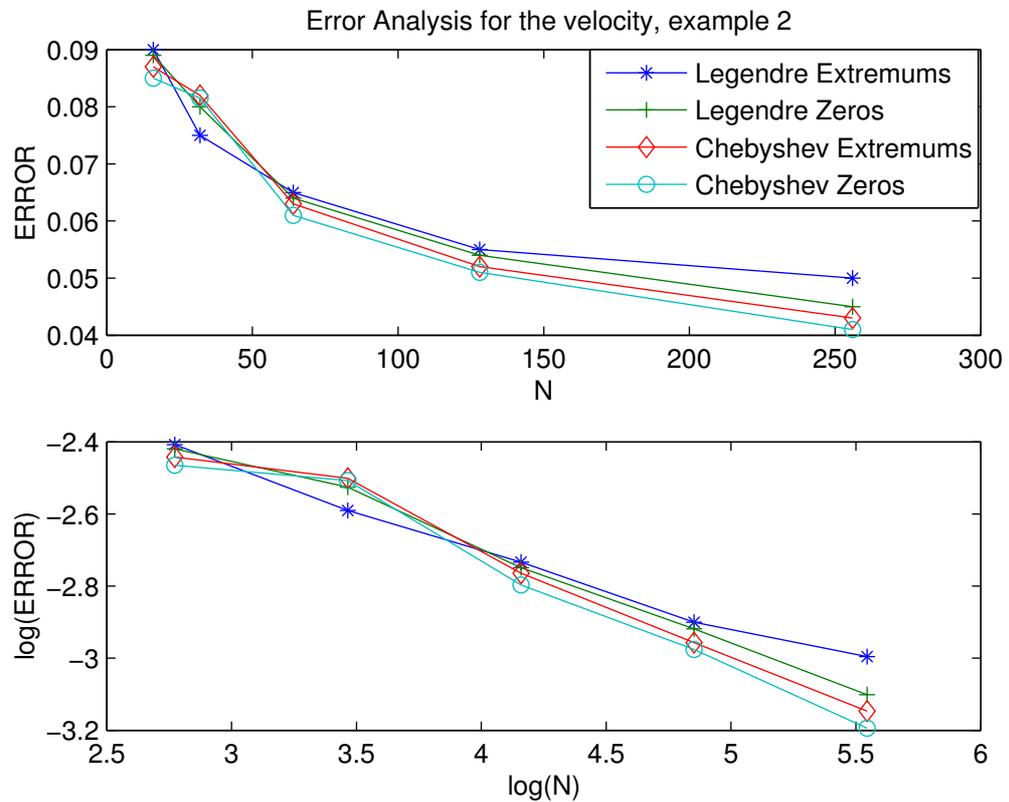


Figure 2. Error Analysis for the velocity, Example 2

It is clear that this technique can be very useful in nonlinear cases as it has been shown above for the Navier Stokes equations. Optimal results were found.

Aknowelegement : We are very grateful for Professor Bernard Philippe for providing the numerical facilities at the IRISA. The authors are also grateful to professor Bernardi for the fruitful discussions during her visit in our department in September 2005. We also wish to thank the referees for their comments and suggestions.

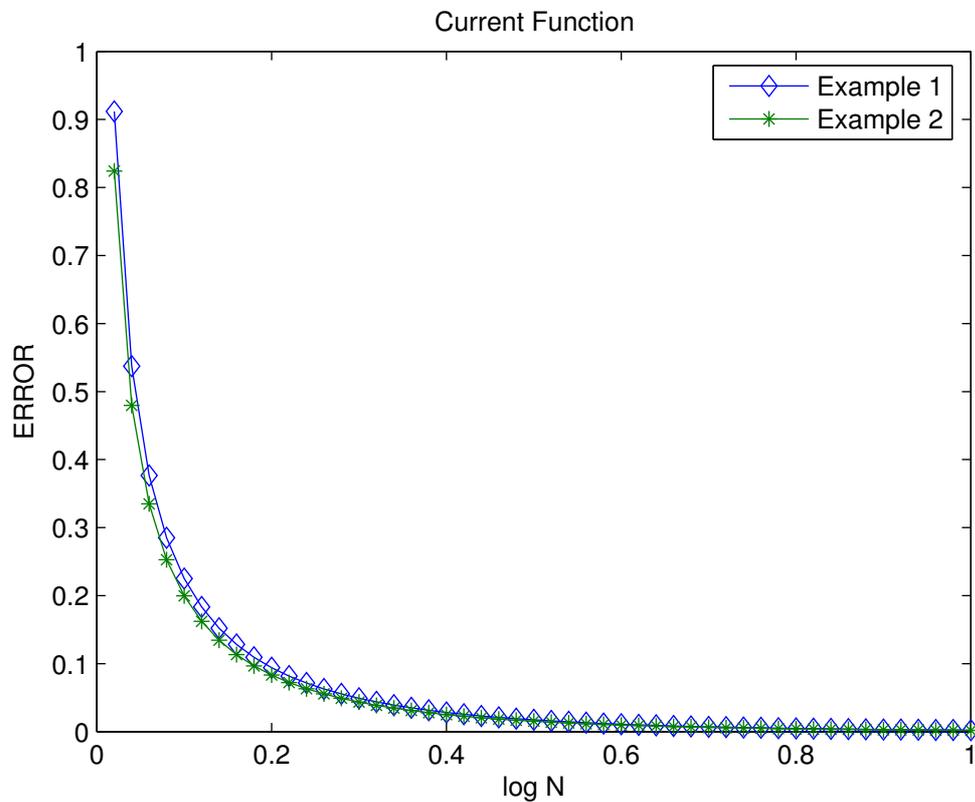


Figure 3. Chebyshev Spectral Error Analysis for the current function

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