Hermite spline interpolants

New methods for constructing and compressing Hermite spline interpolants

Hamid Mraoui* — Driss Sbibih*

* Département de Mathématiques et Informatique
Faculté des Sciences
Université Mohammed I
Oujda - Maroc
hamidmraoui@yahoo.fr; sbibih@sciences.univ-oujda.ac.ma
Research supported in part by PROTARS III, D11/18

ABSTRACT. In this paper, we present a quite simple recursive method for the construction of the classical tensor product Hermite spline interpolant $f_{k,l}$ of a function f defined on $\Omega = [a,b] \times [c,d]$. We show that $f_{k,l}$ can be written as a sum of $f_{k-1,l-1}$ and of particular splines that have interesting properties. As application of this method, we give an algorithm which allows to compress Hermite data. In order to illustrate our results, some numerical examples are presented.

RÉSUMÉ. Dans ce travail, nous présentons une méthode simple permettant de construire le produit tensoriel des interpolants splines d'Hermite $f_{k,l}$ d'une fonction f définie sur $\Omega = [a,b] \times [c,d]$. Nous montrons que $f_{k,l}$ peut être écrit comme somme de $f_{k-1,l-1}$ et de quelques fonctions splines qui vérifient des propriétés intéressantes. Comme application de cette décomposition, nous décrivons un algorithme qui permet de compresser des données d'Hermite. Pour illustrer nos différents résultats, nous donnons quelques exemples numériques.

KEYWORDS : Smoothing of surfaces, tensor product, Hermite spline interpolants, compression of Hermite data.

MOTS-CLÉS : Lissage de surfaces, produit tensoriel, interpolants d'Hermite, compression de données d'Hermite.

1. Introduction

Let $\Omega = [a, b] \times [c, d]$ be a bounded domain of the plane and let $\Delta_{n,m}$ be a rectangular partition of Ω. We denote by $\mathcal{C}^{k,l}(\Omega)$ the space of functions defined on Ω such that their mixed derivatives of order $r + s$, $0 \leq r \leq k$ and $0 \leq s \leq l$, are continuous. Let be given a piecewise function f in $\mathcal{C}^{k,l}(\Omega)$ except at the knots (x_i, y_j) of $\Delta_{n,m}$ where it is only of class $\mathcal{C}^{k_i,l_j}$, $k_i \leq k$ and $l_j \leq l$. In [9], we have proposed a new method which smoothes the function f at (x_i, y_j), $0 \leq i \leq n$ and $0 \leq j \leq m$. More specifically, we have described algorithms allowing to transform f into another function which becomes of class $\mathcal{C}^{k,l}$ on the whole domain Ω . Our aim in this paper is to apply this method for computing recursively the tensor product Hermite spline interpolants, and afterwards to develop an algorithm for compressing Hermite data used in the construction of these interpolants. In other words, if the values and the derivatives up to order k (resp. l) of f in the first variable (resp. second variable) at the knots of $\Delta_{n,m}$ are available, then we show that the Hermite spline interpolant $f_{k,l}$ of f can be decomposed in the form: $f_{k,l} = f_{0,0} + \sum_{r=0}^{k-1} \sum_{s=0}^{l-1} \sum_{t=1}^{3} g_{r,s}^t$, where $f_{0,0}$ is the bilinear tensor product interpolant to f at the vertices of Ω, and $g_{r,s}^t$ are piecewise polynomial functions that satisfy some particular and interesting properties. However, this fact means that $g_{r,s}^t$ can be considered as correction terms or detail functions. Then, according to the above decomposition of $f_{k,l}$, it is natural to make a connection with a multiresolution analysis. In this regard, we achieve compression of data in the standard way by thresholding out small coefficients of detail functions $g_{r,s}^t$, and we show that this process leads to the omission of some Hermite data without affecting the smoothness of the Hermite spline interpolant and sacrificing the quality of approximation.

The paper is organized as follows. In Section 2, we recall the decomposition of univariate Hermite spline interpolants. In Section 3, we give a recursive computation of the tensor product Hermite spline interpolant $f_{k,l}$ and we study the properties of the detail functions $g_{r,s}^t$. Finally, in Section 4, we describe an algorithm allowing to compress Hermite data. All these results are illustrated with some numerical examples.

2. Preliminary results

In this section, we recall some results which we will need in what follows. They are developed in [6] and concern a recursive construction of univariate Hermite interpolants. Let $k \in \mathbf{N}$. For all $p \in \{0, \ldots, k\}$, we define the piecewise polynomials $\phi_{p,i}$ and $\overline{\phi}_{p,i}$ by

$$\begin{aligned} \phi_{p,i}(x) &= \frac{(x-x_i)^p}{p!} \left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^{p+1} \quad \text{if } x \in [x_i, x_{i+1}] \text{ and } 0 \text{ otherwise.} \\ \overline{\phi}_{p,i}(x) &= (-1)^p \phi_{p,i}(x_{i+1}+x_i-x). \end{aligned}$$

It is easy to see that for a fixed i, $\phi_{p,i}$, $0 \le p \le k$, are particular cases of the classical Bernstein basis of the space of polynomials of degree $\le 2p+1$ restricted to the subinterval $[x_i, x_{i+1}]$.

Let f_k be the Hermite interpolant to a function f at the knots x_i, $0 \le i \le n$, of the interval $[a,b]$. According to [6], we have the following result.

Theorem 2.1. *Let $g_{k-1} = f_k - f_{k-1}$. Then we have the following properties.*

(i) g_{k-1} is a piecewise polynomial of class $\mathcal{C}^{k-1}$ and degree $2k+1$ on $[a,b]$, and satisfies

$$g_{k-1}^{(r)}(x_i) = 0, \quad \textit{for} \quad 0 \le r \le k-1 \quad \textit{and} \quad 0 \le i \le n.$$

(ii) g_{k-1} can be expressed only in terms of the basis functions $\phi_{k,i}$ and $\overline{\phi}_{k,i}$, $i \le i \le n-1$,

$$g_{k-1}(x) = \sum_{i=0}^{n-1} \left(\delta_i^k \phi_{k,i}(x) + \overline{\delta}_i^k \overline{\phi}_{k,i}(x)\right),$$

$$\textit{where } \delta_i^k = \frac{k!}{h_i^k} \sum_{r=0}^{k} \frac{h_i^r}{r!} \left[\binom{2k-r-l}{k-1} f^{(r)}(x_i) - (-1)^r \binom{2k-r-l}{k} f^{(r)}(x_{i+1})\right],$$

$$\textit{and } \overline{\delta}_i^k = (-1)^k \frac{k!}{h_i^k} \sum_{r=0}^{k} \frac{h_i^r}{r!} \left[(-1)^r \binom{2k-r-l}{k-1} f^{(r)}(x_{i+1}) - \binom{2k-r-l}{k} f^{(r)}(x_i)\right], \quad h_i = x_{i+1} - x_i.$$

REMARK. — Using the fact that $f_k = f_{k-1} + g_{k-1}$, we can easily show that the coefficients δ_i^k and $\overline{\delta}_i^k$ can be written in form

$\delta_i^k = f^{(k)}(x_i) - f_{k-1}^{(k)}(x_i^+)$ and $\overline{\delta}_i^k = f^{(k)}(x_i) - f_{k-1}^{(k)}(x_{i+1}^-)$. Then, we deduce that

$$\begin{aligned} f_{k-1}^{(k)}(x_i^+) &= -\frac{k!}{h_i^k} \sum_{r=0}^{k-1} \frac{h_i^r}{r!} \left[\binom{2k-r-l}{k-1} f^{(r)}(x_i) - (-1)^r \binom{2k-r-l}{k} f^{(r)}(x_{i+1})\right], \\ f_{k-1}^{(k)}(x_i^-) &= (-1)^k \frac{k!}{h_i^k} \sum_{r=0}^{k-1} \frac{h_i^r}{r!} \left[(-1)^r \binom{2k-r-l}{k-1} f^{(r)}(x_{i-1}) - \binom{2k-r-l}{k} f^{(r)}(x_i)\right]. \end{aligned}$$

3. Recursive computation of Hermite spline interpolants

Let $\Delta_n = \{a = x_0 < x_1 < \ldots < x_m = b\}$ (resp. $\Delta_m = \{c = y_0 < y_1 < \ldots < y_m = d\}$) be a partition of $I = [a, b]$ (resp. $J = [c, d]$) and let

$$\Delta_{n,m} = \{R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}],\ \ 0 \le i \le n-1 \ \ \text{et}\ \ 0 \le j \le m-1\}$$

be a partition of $\Omega = I \times J$. We denote by

$$h_i = x_{i+1} - x_i,\ h = \max_{0\le i\le n-1} h_i,\ \hbar_j = y_{j+1} - y_j,\ \hbar = \max_{0\le j\le m-1} \hbar_j,$$

$$\begin{aligned} V_k &= \mathbb{P}^k_{2k+1}(I, \Delta_n) = \{g \in \mathcal{C}^k(I) :\ \ g_{|[x_i,x_{i+1}]} \in \mathbb{P}_{2k+1}\}, \\ V_l &= \mathbb{P}^l_{2l+1}(J, \Delta_m) = \{g \in \mathcal{C}^l(J) :\ \ g_{|[y_j,y_{j+1}]} \in \mathbb{P}_{2l+1}\}, \end{aligned}$$

and by $E_{k,l}$ the tensor product space defined by

$$E_{k,l} = V_k \otimes V_l = \{f :\ \Omega \to \mathbb{R} :\ f(.,y) \in \mathcal{C}^k(I),\ f(x,.) \in \mathcal{C}^l(J) \text{ and } f_{|R_{i,j}} \in \mathbb{P}_{2k+1,2l+1}\},$$

where $\mathbb{P}_{2k+1,2l+1}$ is the space of bivariate polynomials of degree $\le 2k+1$ in the first variable and of degree $\le 2l+1$ in the second variable.

Let $f_{k,l} \in E_{k,l}$ be the tensor product Hermite spline interpolant of a function f, defined on Ω, at vertices (x_i, y_j), $0 \le i \le n$ and $0 \le j \le m$, of the partition $\Delta_{n,m}$. If we denote $D^{(r,s)}f = \frac{\partial^{r+s} f}{\partial x^r \partial y^s}$, then it is well known, see [2], that $f_{k,l}$ is uniquely determined by the following interpolation conditions

$$D^{(r,s)} f_{k,l}(x_i, y_j) = D^{(r,s)} f(x_i, y_j),\ \ 0 \le i \le n,\ \ 0 \le j \le m,\ \ 0 \le r \le k,\ \ 0 \le s \le l.$$

Using the classical Hermite basis functions $\Psi^{i,j}_{k,l,r,s} = \psi^r_{k,i} \otimes \psi^s_{l,j}$, $(i,j) \in \{0, \ldots, n\} \times \{0, \ldots, m\}$ and $(r,s) \in \{0, \ldots, k\} \times \{0, \ldots, l\}$, of the space $E_{k,l}$, the interpolant $f_{k,l}$ can be written in the form

$$f_{k,l}(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{r=0}^{k} \sum_{s=0}^{l} D^{(r,s)} f(x_i, y_j)\ \Psi^{i,j}_{k,l,r,s}(x,y), \tag{1}$$

where $\{\psi^r_{k,i},\ 0 \le r \le k,\ 0 \le i \le n\}$ (resp. $\{\psi^s_{l,j},\ 0 \le s \le l,\ 0 \le j \le m\}$) is the classical Hermite basis for V_k (resp. V_l).

In spite of the simplicity of (1), the lack of recursive formulae for computing the basis functions $\Psi^{i,j}_{k,l,r,s}$ makes its use rather complicated. In order to remedy this problem, we present a method, based on the results given in [9], which allows to compute recursively the spline $f_{k,l}$. Indeed, with the help of the smoothing algorithms described in [9], we transform the piecewise bilinear tensor product interpolant to f at the vertices of Ω into the Hermite interpolant $f_{k,l}$ by adding to $f_{0,0}$ some appropriate functions. More specifically, if we put

$$\begin{aligned}
g^1_{k-1,l-1}(x,y) &= \sum_{i=0}^{n-1}\sum_{j=0}^{m}\sum_{s=0}^{l-1}\left(\delta^{k,s}_{i,j}\ \phi_{k,i}(x)+\overline{\delta}^{k,s}_{i,j}\ \overline{\phi}_{k,i}(x)\right)\psi^s_{l-1,j}(y),\\
g^2_{k-1,l-1}(x,y) &= \sum_{j=0}^{m-1}\sum_{i=0}^{n}\sum_{r=0}^{k-1}\left(\delta^{r,l}_{i,j}\ \phi_{l,j}(y)+\overline{\delta}^{r,l}_{i,j}\ \overline{\phi}_{l,j}(y)\right)\psi^r_{k-1,i}(x),\\
g^3_{k-1,l-1}(x,y) &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\delta^1_{i,j,k,l}\phi_{k,i}(x)\phi_{l,j}(y)+\delta^2_{i,j,k,l}\overline{\phi}_{k,i}(x)\phi_{l,j}(y)+\right.\\
&\qquad \left.\delta^3_{i,j,k,l}\phi_{k,i}(x)\overline{\phi}_{l,j}(y)+\delta^4_{i,j,k,l}\overline{\phi}_{k,i}(x)\overline{\phi}_{l,j}(y)\right],
\end{aligned}$$

where

$$\begin{aligned}
\delta^{k,s}_{i,j} &= D^{(k,s)}f(x_i,y_j)-D^{(k,s)}f_{k-1,l-1}(x_i^+,y_j),\\
\overline{\delta}^{k,s}_{i,j} &= D^{(k,s)}f(x_{i+1},y_j)-D^{(k,s)}f_{k-1,l-1}(x_{i+1}^-,y_j),\\
\delta^{r,l}_{i,j} &= D^{(r,l)}f(x_i,y_j)-D^{(r,l)}f_{k-1,l-1}(x_i,y_j^+),\\
\overline{\delta}^{r,l}_{i,j} &= D^{(r,l)}f(x_i,y_{j+1})-D^{(r,l)}f_{k-1,l-1}(x_i,y_{j+1}^-),
\end{aligned}$$

and

$$\begin{aligned}
\delta^1_{i,j,k,l} &= \Big(D^{(k,l)}f(x_i,y_j)-D^{(k,l)}f_{k,l-1}(x_i,y_j^+)-D^{(k,l)}f_{k-1,l}(x_i^+,y_j)\\
&\quad +D^{(k,l)}f_{k-1,l-1}(x_i^+,y_j^+)\Big),\\
\delta^2_{i,j,k,l} &= \Big(D^{(k,l)}f(x_i,y_j)-D^{(k,l)}f_{k,l-1}(x_i,y_j^+)-D^{(k,l)}f_{k-1,l}(x_{i+1}^-,y_j)\\
&\quad +D^{(k,l)}f_{k-1,l-1}(x_{i+1}^-,y_j^+)\Big),\\
\delta^3_{i,j,k,l} &= \Big(D^{(k,l)}f(x_i,y_j)-D^{(k,l)}f_{k,l-1}(x_i,y_{j+1}^-)-D^{(k,l)}f_{k-1,l}(x_i^+,y_j)\\
&\quad +D^{(k,l)}f_{k-1,l-1}(x_i^+,y_{j+1}^-)\Big),
\end{aligned}$$

$$\delta^4_{i,j,k,l} = \Big(D^{(k,l)}(x_i,y_j) - D^{(k,l)}f_{k,l-1}(x_i,y^-_{j+1}) - D^{(k,l)}f_{k-1,l}(x^-_{i+1},y_j) \\ +D^{(k,l)}f_{k-1,l-1}(x^-_{i+1},y^-_{j+1})\Big).$$

Then we have the following result.

Theorem 3.1. *The Hermite spline interpolants $f_{k,l-1}$, $f_{k-1,l}$ and $f_{k,l}$ can be decomposed in the form*

$$f_{k,l-1} = f_{k-1,l-1} + g^1_{k-1,l-1}; \;\; f_{k-1,l} = f_{k-1,l-1} + g^2_{k-1,l-1};$$
$$f_{k,l} = f_{k-1,l-1} + g^1_{k-1,l-1} + g^2_{k-1,l-1} + g^3_{k-1,l-1}.$$

Proof. Let $\widetilde{f}_{k,l-1} = f_{k-1,l-1} + g^1_{k-1,l-1}$. Remark that $\widetilde{f}_{k,l-1}$ can be written in the form

$$\widetilde{f}_{k,l-1} = f_{k-1,l-1} + \sum_{i=0}^{n-1}\Big(\delta^k_i(y)\phi_{k,i}(x) + \overline{\delta}^k_i(y)\overline{\phi}_{k,i}(x)\Big), \;\; \text{where}$$

$$\delta^k_i(y) = \overline{f}^{(k)}_{l-1}(x_i,y) - \frac{\partial^k}{\partial x^k} f_{k-1,l-1}(x^+_i,y),$$
$$\overline{\delta}^k_i(y) = \overline{f}^{(k)}_{l-1}(x_{i+1},y) - \frac{\partial^k}{\partial x^k} f_{k-1,l-1}(x^-_{i+1},y),$$

and $\overline{f}^{(k)}_{l-1}(x_i,.) = \sum_{j=0}^{m}\sum_{s=0}^{l-1} D^{(k,s)}f(x_i,y_j)\,\psi^s_{l-1,j}$.

Using the fact that $(\phi_{k,i})^{(r)}(x^+_l) = (\overline{\phi}_{k,i})^{(r)}(x^-_{l+1}) = \delta_{k,r}\delta_{i,l}$, for $0 \le r \le k$, we obtain

$$\frac{\partial^k}{\partial x^k}\widetilde{f}_{k,l-1}(x^+_i,y) = \overline{f}^{(k)}_{l-1}(x_i,y) \;\text{ and }\; \frac{\partial^k}{\partial x^k}\widetilde{f}_{k,l-1}(x^-_i,y) = \overline{f}^{(k)}_{l-1}(x_i,y).$$

Thus, $\widetilde{f}_{k,l-1}(.,y)$ is of class $\mathcal{C}^k$ at x_i. As $f_{k-1,l-1}(.,y) \in V_{k-1}$, $\phi_{k,i}$ and $\overline{\phi}_{k,i}$, $0 \le i \le n$, are functions in $\mathbf{P}^{k-1}_{2k+1}(I,\Delta_n)$, we deduce that $\widetilde{f}_{k,l-1}(.,y) \in V_k$, for each $y \in J$. Consequently $\widetilde{f}_{k,l-1} \in E_{k,l-1}$.
On the other hand, it is easy to verify that

$$D^{(r,s)}\widetilde{f}_{k,l-1}(x_i,y_j) = D^{(r,s)}f(x_i,y_j),\; \forall\, 0 \le i \le n,\; 0 \le r \le k,\; 0 \le j \le m,\; 0 \le s \le l-1.$$

Then, according to the uniqueness of the Hermite interpolant, we get $\widetilde{f}_{k,l-1} = f_{k,l-1}$. In the same way, we can prove the decomposition of $f_{k-1,l}$ and $f_{k,l}$. ■

For the implementation of the above decomposition method, it is necessary to express all the coefficients that appear in this decomposition in terms of Hermite data. Then, we have the following result.

Proposition 3.2.

$$\begin{cases} \delta_{i,j}^{k,s} = \frac{k!}{h_i^k}\displaystyle\sum_{r=0}^{k}\frac{h_i^r}{r!}\left[\binom{2k-r-l}{k-1}D^{(r,s)}f(x_i,y_j)-(-1)^r\binom{2k-r-l}{k}D^{(r,s)}f(x_{i+1},y_j)\right], \\ \overline{\delta}_{i,j}^{k,s} = (-1)^k\frac{k!}{h_i^k}\displaystyle\sum_{r=0}^{k}\frac{h_i^r}{r!}\left[(-1)^r\binom{2k-r-l}{k-1}D^{(r,s)}f(x_{i+1},y_j)-\binom{2k-r-l}{k}D^{(r,s)}f(x_i,y_j)\right], \\ \delta_{i,j}^{r,l} = \frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[\binom{2l-s-l}{l-1}D^{(r,s)}f(x_i,y_j)-(-1)^s\binom{2l-s-l}{l}D^{(r,s)}f(x_i,y_{j+1})\right], \\ \overline{\delta}_{i,j}^{r,l} = (-1)^l\frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[(-1)^s\binom{2l-s-l}{l-1}D^{(r,s)}f(x_i,y_{j+1})-\binom{2l-s-l}{l}D^{(r,s)}f(x_i,y_j)\right], \end{cases}$$

and

$$\begin{cases} \delta_{i,j,k,l}^{1} = \frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[\binom{2l-s-l}{l-1}\delta_{i,j}^{k,s}-(-1)^s\binom{2l-s-l}{l}\delta_{i,j+1}^{k,s}\right], \\ \delta_{i,j,k,l}^{2} = \frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[\binom{2l-s-l}{l-1}\overline{\delta}_{i,j}^{k,s}-(-1)^s\binom{2l-s-l}{l}\overline{\delta}_{i,j+1}^{k,s}\right], \\ \delta_{i,j,k,l}^{3} = (-1)^l\frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[(-1)^s\binom{2l-s-l}{l-1}\delta_{i,j+1}^{k,s}-\binom{2l-s-l}{l}\delta_{i,j}^{k,s}\right], \\ \delta_{i,j,k,l}^{4} = (-1)^l\frac{l!}{\hbar_j^l}\displaystyle\sum_{s=0}^{l}\frac{\hbar_j^s}{s!}\left[(-1)^s\binom{2l-s-l}{l-1}\overline{\delta}_{i,j+1}^{k,s}-\binom{2l-s-l}{l}\overline{\delta}_{i,j}^{k,s}\right]. \end{cases}$$

Proof. Using Theorem 2.1, we can prove that

$$D^{(0,s)}f_{k,l}(x,y_j) = D^{(0,s)}f_{k-1,l}(x,y_j) + \sum_{i=0}^{n-1}\left(\delta_{i,j}^{k,s}\phi_{k,i}(x)+\overline{\delta}_{i,j}^{k,s}\overline{\phi}_{k,i}(x)\right),$$

$$D^{(r,0)}f_{k,l}(x_i,y) = D^{(r,0)}f_{k,l-1}(x_i,y) + \sum_{j=0}^{m-1}\left(\delta_{i,j}^{r,l}\,\phi_{l,j}(y)+\overline{\delta}_{i,j}^{r,l}\,\overline{\phi}_{l,j}(y)\right),$$

$$D^{(k,0)}f_{k,l}(x_i,y) - D^{(k,0)}f_{k-1,l}(x_i^+,y) = D^{(k,0)}f_{k,l-1}(x_i,y) \\ -D^{(k,0)}f_{k-1,l-1}(x_i^+,y) + \sum_{j=0}^{m-1}\left(\delta_{i,j,k,l}^{1}\,\phi_{l,j}(y)+\delta_{i,j,k,l}^{3}\,\overline{\phi}_{l,j}(y)\right),$$

$$D^{(k,0)}f_{k,l}(x_{i+1},y) - D^{(k,0)}f_{k-1,l}(x_{i+1}^-,y) = D^{(k,0)}f_{k,l-1}(x_{i+1},y) \\ -D^{(k,0)}f_{k-1,l-1}(x_{i+1}^-,y) + \sum_{j=0}^{m-1}\left(\delta_{i,j,k,l}^{2}\,\phi_{l,j}(y)+\delta_{i,j,k,l}^{4}\,\overline{\phi}_{l,j}(y)\right),$$

and the claim follows from the remark given in Section 2. ∎

Now, for giving an error bound between $f_{k,l}$ and $f_{k-u,l-v}$, with $0 \leq u, v \leq 1$ and $(u,v) \neq (0,0)$, we need the following result.

Lemma 3.3. *Assume that* $f \in \mathcal{C}^{k,l}(\Omega)$. *Let* $c_k = \sum_{r=0}^{[\frac{k-1}{2}]} \binom{k-1}{2r} \binom{2k-2r-1}{k-2r}$ *and* $c_l = \sum_{s=0}^{[\frac{l-1}{2}]} \binom{l-1}{2s} \binom{2l-2s-1}{l-2s}$. *Then, for* $0 \leq j \leq m$ *and* $0 \leq s \leq l-1$, *we have*

$$\delta_{i,j}^{k,s} \leq c_k \, \omega \left(D^{(k,s)} f(.,y_j), h_i \right) \quad \text{and} \quad \overline{\delta}_{i,j}^{k,s} \leq c_k \, \omega \left(D^{(k,s)} f(.,y_j), h_i \right),$$

and for $0 \leq i \leq n$ *and* $0 \leq r \leq k-1$ *we have*

$$\delta_{i,j}^{r,l} \leq c_l \, \omega \left(D^{(r,l)} f(x_i,.), \hbar_j \right) \quad \text{and} \quad \overline{\delta}_{i,j}^{r,l} \leq c_l \, \omega \left(D^{(r,l)} f(x_i,.), \hbar_j \right).$$

Moreover, for $(i,j) \in \{0,\ldots,n-1\} \times \{0,\ldots,m-1\}$

$$\delta_{i,j,k,l}^{t} \leq 2 c_k \, c_l \, \overline{\omega} \left(D^{(k,l)} f, h_i, \hbar_j \right), \quad t \in \{1,2,3,4\}, \quad \text{where}$$

$\omega(g,.)$ *is the modulus of continuity of* g, $g \in \mathcal{C}([a,b])$, *and*

$$\begin{aligned} \omega(D^{(p,q)} f, h_i, \hbar_j) &= \sup_{\|(\alpha,\beta)-(\alpha',\beta')\| \leq (h_i,\hbar_j)} \left\{ |D^{(p,q)} f(\alpha,\beta) - D^{(p,q)} f(\alpha',\beta')| \right\}, \\ \overline{\omega}(D^{k,l} f, h_i, \hbar_j) &= \sup \left\{ \omega(D^{(p,q)} f, h_i, \hbar_j) : \ (p,q) \in \{0,\ldots,k\} \times \{0,\ldots,l\} \right\}. \end{aligned}$$

Proof. From Proposition 3.2, and using the Taylor expansions of $D^{(r,s)} f(x_{i+1}, y_j)$, $0 \leq r \leq k$, at x_i we obtain

$$\begin{aligned} \delta_{i,j}^{k,s} &= D^{(k,s)} f(x_i,y_j) + \sum_{r=0}^{k-1} \alpha_{i,r}^{k} D^{(r,s)} f(x_i,y_j) \\ &\quad - \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \binom{2k-r-1}{k-r} D^{(k,s)} f(\zeta_{i,r}, y_j), \end{aligned} \tag{2}$$

where $x_i \leq \zeta_{i,r} \leq x_{i+1}$ and $\alpha_{i,r}^{k} = \frac{h_i^{r-k} k(2k-r-1)!}{r!(k-r)!} - \sum_{p=0}^{r} (-1)^p \frac{h_i^{p-k}(2k-p-1)!}{p!(k-p-1)!(r-p)!}$.

With the help of the identity $\sum_{r=0}^{r_1} (-1)^r \binom{r_1}{r} \binom{p-r}{q} = \binom{p-r_1}{q-r_1}$, for all $p \geq q \geq r_1$, (see [13], p.13), we deduce on the one hand that $\alpha^k_{i,r} = 0$, for all $r = 0, \ldots, k-1$, and on the other hand that $\sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \binom{2k-r-1}{k-r} = 1$. Hence, (2) becomes

$$\begin{aligned}\delta^{k,s}_{i,j} &= D^{(k,s)} f(x_i, y_j) + \sum_{r=0}^{[\frac{k}{2}]-1} \binom{k-1}{2r+1} \binom{2k-2r-2}{k-1} D^{(k,s)} f(\zeta_{i,2r+1}, y_j) \\ &\quad - \sum_{r=0}^{[\frac{k-1}{2}]} \binom{k-1}{2r} \binom{2k-2r-2}{k-1} D^{(k,s)} f(\zeta_{i,2r}, y_j). \qquad (3)\end{aligned}$$

Since $1 + \sum_{r=0}^{[\frac{k}{2}]-1} \binom{k-1}{2r+1} \binom{2k-2r-2}{k-1} = \sum_{r=0}^{[\frac{k-1}{2}]} \binom{k-1}{2r} \binom{2k-2r-2}{k-1}$, and

$$\left| D^{(k,s)} f(\zeta_{i,2r}, y_j) - D^{(k,s)} f(\zeta_{i,2r'+1}, y_j) \right| \leq \omega(D^{(k,s)} f(., y_j), h_i)$$

for all $0 \leq r \leq [\frac{k-1}{2}]$ and $0 \leq r' \leq [\frac{k}{2}] - 1$, (3) can be written as a sum of c_k terms of the form $\left(D^{(k,s)} f(\zeta_{i,2r}, y_j) - D^{(k,s)} f(\zeta_{i,2r'+1}, y_j) \right)$. Therefore, we have

$$\delta^{k,s}_{i,j} \leq c_k \, \omega(D^{(k,s)} f(., y_j), h_i).$$

Using a similar technique, one can show the other inequalities of the Lemma. ■

Proposition 3.4. *If we denote by $T^t_{k-1,l-1}$ the operator satisfying $T^t_{k-1,l-1} f = g^t_{k-1,l-1}$, $1 \leq t \leq 3$, then $T^t_{k-1,l-1} p = 0$ for all $p \in \mathbb{P}_{2k-1,2l-1}$. Moreover, we have*

$$\|g^1_{k-1,l-1}\|_\infty \leq \frac{c_k h^k}{k! 4^k} \sigma_{l-1} \, \overline{\omega}(D^{(k,l)} f, h, \hbar), \quad \|g^2_{k-1,l-1}\|_\infty \leq \frac{c_l \hbar^l}{l! 4^l} \sigma_{k-1} \, \overline{\omega}(D^{(k,l)} f, h, \hbar),$$

$$\|g^3_{k-1,l-1}\|_\infty \leq \frac{2 c_k c_l h^k \hbar^l}{k! l! 4^{k+l}} \, \overline{\omega}(D^{k+l} f, h, \hbar),$$

where $\sigma_{l-1} = \sum_{s=0}^{l-1} \sum_{j=0}^{m} \|\psi^s_{l-1,j}\|_{\infty,J}$ *and* $\sigma_{k-1} = \sum_{r=0}^{k-1} \sum_{i=0}^{n} \|\psi^r_{k-1,i}\|_{\infty,I}$.

Proof. Let $H_{k-1,l}$, $H_{k,l-1}$ and $H_{k,l}$ be the Hermite interpolants to f at knots (x_i, y_j), i.e., $H_{k-1,l} f = f_{k-1,l}$, $H_{k,l-1} f = f_{k,l-1}$ and $H_{k,l} f = f_{k,l}$. According to the decomposition of $f_{k-1,l}$, $f_{k,l-1}$ and $f_{k,l}$ given in Theorem 3.1, we can write

$$\begin{aligned} H_{k,l-1} &= H_{k-1,l-1} + T^1_{k-1,l-1}, \; H_{k-1,l} = H_{k-1,l-1} + T^2_{k-1,l-1}, \\ H_{k,l} &= H_{k-1,l-1} + T^1_{k-1,l-1} + T^2_{k-1,l-1} + T^3_{k-1,l-1}. \end{aligned}$$

As $H_{k,l-1}q = q$ for all $q \in \mathbb{P}_{2k+1,2l-1}$, $H_{k-1,l}q = q$ for all $q \in \mathbb{P}_{2k-1,2l+1}$ and $H_{k,l}q = q$ for all $q \in \mathbb{P}_{2k+1,2l+1}$, we deduce that $T^t_{k-1,l-1}p = 0$ for all $p \in \mathbb{P}_{2k-1,2l-1}$ and $t \in \{1,2,3\}$.
On the other hand, for all $x \in [x_i, x_{i+1}]$, $0 \leq i \leq n-1$, we have

$$g^1_{k-,l-1}(x,y) = \frac{1}{k!}(x-x_i)^k\,(\omega_i(x))^k \times \sum_{s=0}^{l-1}\sum_{j=0}^{m}\left[\omega_i(x)\delta^{k,s}_{i,j} + (-1)^k\,(1-\omega_i(x))\,\overline{\delta}^{k,s}_{i,j}\right]\psi^s_{l-1,j}(y),$$

where $\omega_i(x) = \left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)$.
Since $0 \leq (x-x_i)\,\omega_i(x) \leq \frac{h_i}{4}$, we deduce from Lemma 3.3 that

$$\|g^1_{k-1,l-1}\|_\infty \leq \frac{c_k h^k}{k!4^k}\sigma_{l-1}\,\overline{\omega}(D^{(k,l)}f, h, \hbar).$$

Using a similar technique one can prove that

$$\|g^2_{k-1,l-1}\|_\infty \leq \frac{c_l \hbar^l}{l!4^l}\sigma_{k-1}\,\overline{\omega}\left(D^{(k,l)}f, h, \hbar\right).$$

Concerning $g^3_{k-1,l-1}$, we have for all $(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$

$$\begin{aligned} g^3_{k-1,l-1}(x,y) = {} & \frac{1}{k!l!}(x-x_i)^k\,(\omega_i(x))^k\,(y-y_j)^l\,(\omega_j(y))^l\,\big[\omega_i(x)\,\omega_j(y)\delta^1_{i,j,k,l} + \\ & (-1)^k(1-\omega_i(x))\,\omega_j(y)\delta^2_{i,j,k,l} + (-1)^l\omega_i(x)\,(1-\omega_j(y))\delta^3_{i,j,k,l} + \\ & (-1)^{k+l}(1-\omega_i(x))\,(1-\omega_j(y))\delta^4_{i,j,k,l}\big]\,. \end{aligned}$$

Then, using once again Lemma 3.3, we get

$$\|g^3_{k-1,l-1}\|_\infty \leq \frac{2c_k c_l h^k \hbar^l}{k!l!4^{k+l}}\,\overline{\omega}(D^{k+l}f, h, \hbar).$$

■

3.1. Numerical example

In this example, we describe the decomposition of the tensor product Hermite spline interpolant $f_{2,2}$ that interpolates the values and the derivatives of the function $f(x,y) = \cos(\frac{x^2-y}{2})$, defined on $\Omega = [0,4] \times [-2,2]$, at vertices $(x_i, y_j) = (i,j)$ of $\Omega \cap \mathbb{Z}^2$. According to Section 3, we have $f_{2,2} = f_{0,0} + g^1_{0,0} + g^2_{0,0} + g^3_{0,0} + g^1_{1,1} + g^2_{1,1} + g^3_{1,1}$. In Figure 1, we give the graphs of f, $f_{2,2}$, $f_{0,0}$ and some detail functions.

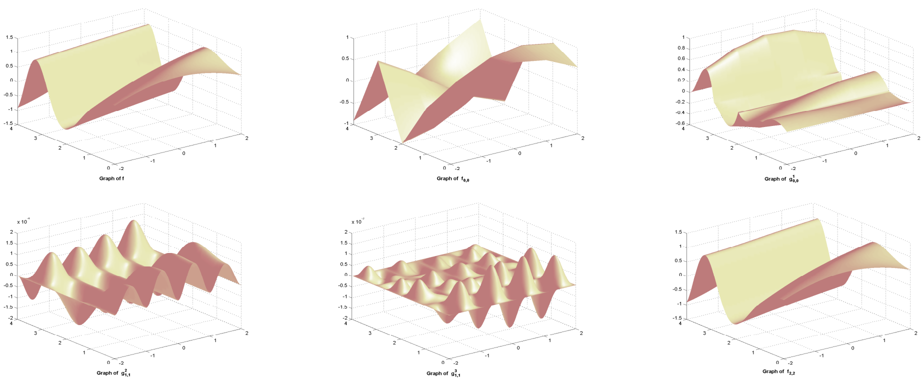

Figure 1. *Decomposition of* $f_{2,2}$.

4. Compressing data related to Hermite spline interpolants

The above decomposition of $f_{k,l}$ can be regarded as an example of a Faber interpolation scheme, see [1] and [3]. It gives rise to new hierarchical bases for the spaces $E_{r,s}$, $0 \le r \le k$ and $0 \le s \le l$, which can be used as tools for solving several mathematical problems like those studied in [10], [11], [12] and [14]. Moreover, in view of the multiresolution structure of this decomposition, the added functions $g^t_{r,s}$ can be considered as correction terms. Hence, as in the wavelet theory, a data compression can be achieved by thresholding out the small coefficients of $g^t_{r,s}$, $1 \le t \le 3$, i.e., removing the coefficients that are less than a given threshold value $\epsilon_{k,l}$. From the expressions of these coefficients, given in Section 3, the removal of one of them is equivalent to substitute the data $D^{(r,s)}f(x_i, y_j)$ for a linear combination of $D^{(r,s)}f_{r-u,s-v}(x_i^{\pm}, y_j^{\pm})$, $0 \le u, v \le 1$ and $(u, v) \ne (0, 0)$. Therefore, this method allows us to omit some Hermite data without sacrificing the quality of approximation. A data Hermite compression method has been developed in [6] for the univariate case, and the obtained results are interesting. In this section, we generalize this method to the bivariate case.

In what follows, we give an algorithm that compresses $f_{k,l}$. This algorithm provides the coefficients $\left\{(\delta^{r,s_1}_{i,j})^c,\ 0 \le i \le n-1,\ 0 \le j \le m \ and\ 1 \le s_1 \le s-1\right\}$ of correction terms $(g^1_{r-1,s-1})^c$, $\left\{(\delta^{r_1,s}_{i,j})^c,\ 0 \le i \le n,\ 0 \le j \le m-1 \ and\ 1 \le r_1 \le r-1\right\}$ of correction terms $(g^2_{r-1,s-1})^c$ and $\left\{(\delta^t_{i,j,k,l})^c,\ 1 \le t \le 4,\ 1 \le i \le n-1 \ and\ 1 \le j \le m-1\right\}$ of correction terms $(g^3_{r-1,s-1})^c$, $1 \le r \le k$ and $1 \le s \le l$, which are added to $f_{0,0}$ in order to obtain the corresponding compressed interpolant $f^c_{k,l}$.

Algorithm 4.1

Step 1

For $j = 0, \dots, m$,

For $s = 1, \dots, l-1$, do

For $i = 1, \dots, n-1$, let $\Gamma_{i,j}^{k,s} = min\left(|\delta_{i,j}^{k,s}|, |\overline{\delta}_{i-1,j}^{k,s}|\right)$

first case : $\Gamma_{i,j}^{k,s} \leq \epsilon_{k,l}$

i) $\Gamma_{i,j}^{k,s} = |\delta_{i,j}^{k,s}|$, put $(\delta_{i,j}^{k,s})^c = 0$ and

$$(\overline{\delta}_{i-1,j}^{k,s})^c = D^{(k,s)} f_{k,l}(x_i^+, y_j) - D^{(k,s)} f_{k-1,l}(x_i^-, y_j)$$
$$= \overline{\delta}_{i-1,j}^{k,s} - \delta_{i,j}^{k,s}$$

ii) $\Gamma_{i,j}^{k,s} = \overline{\delta}_{i-1,j}^{k,s}$, put $(\overline{\delta}_{i-1,j}^{k,s})^c = 0$ and

$$(\delta_{i,j}^{k,s})^c = D^{(k,s)} f_{k,l}(x_i^-, y_j) - D^{(k,s)} f_{k-1,l}(x_i^+, y_j)$$
$$= \delta_{i,j}^{k,s} - \overline{\delta}_{i-1,j}^{k,s}$$

second case : $\Gamma_{i,j}^{k,s} > \epsilon_{k,l}$

put $(\delta_{i,j}^{k,s})^c = \delta_{i,j}^{k,s}$ and $(\overline{\delta}_{i-1,j}^{k,s})^c = \overline{\delta}_{i-1,j}^{k,s}$

End(i)

End(s)

End(j).

Step 2

For $i = 0, \dots, n$,

For $r = 1, \dots, k-1$, do

For $j = 1, \dots, m-1$, let $\Gamma_{i,j}^{r,l} = min\left(|\delta_{i,j}^{r,l}|, |\overline{\delta}_{i,j-1}^{r,l}|\right)$

first case : $\Gamma_{i,j}^{r,l} \leq \epsilon_{k,l}$

i) $\Gamma_{i,j}^{r,l} = |\delta_{i,j}^{r,l}|$, put $(\delta_{i,j}^{r,l})^c = 0$ and

$$(\overline{\delta}_{i,j-1}^{r,l})^c = D^{(r,l)} f_{k,l}(x_i, y_j^+) - D^{(r,l)} f_{k,l-1}(x_i, y_j^-)$$
$$= \overline{\delta}_{i,j-1}^{r,l} - \delta_{i,j}^{r,l}$$

ii) $\Gamma_{i,j}^{r,l} = \overline{\delta}_{i,j-1}^{r,l}$, put $(\overline{\delta}_{i,j-1}^{r,l})^c = 0$ et

$$(\delta_{i,j}^{r,l})^c = D^{(r,l)} f_{k,l}(x_i, y_j^-) - D^{(r,l)} f_{k,l-1}(x_i, y_j^+)$$
$$= \delta_{i,j}^{r,l} - \overline{\delta}_{i,j-1}^{r,l}$$

second case : $\Gamma_{i,j}^{r,l} > \epsilon_{k,l}$

put $(\delta_{i,j}^{r,l})^c = \delta_{i,j}^{r,l}$ and $(\overline{\delta}_{i,j-1}^{r,l})^c = \overline{\delta}_{i,j-1}^{r,l}$

End(j)

End(r)

End(i).

Step 3

For $i = 0, \ldots, n$,
  For $j = 0, \ldots, m$,, do
    let $\Gamma_{i,j}^{k,l} = min\left(|\delta^1_{i,j,k,l}|, |\delta^2_{i-1,j,k,l}|, |\delta^3_{i,j-1,k,l}|, |\delta^4_{i-1,j-1,k,l}|\right)$
      first case : $\Gamma_{i,j}^{k,l} \leq \epsilon_{k,l}$
        i) $\Gamma_{i,j}^{k,l} = |\delta^1_{i,j,k,l}|$, put $(\delta^1_{i,j,k,l})^c = 0,$ and
          $(\delta^2_{i-1,j,k,l})^c = \delta^2_{i-1,j,k,l} - \delta^1_{i,j,k,l}$
          $(\delta^3_{i,j-1,k,l})^c = \delta^3_{i,j-1,k,l} - \delta^1_{i,j,k,l}$
          $(\delta^4_{i-1,j-1,k,l})^c = \delta^4_{i-1,j-1,k,l} - \delta^1_{i,j,k,l}$
        ii) $\Gamma_{i,j}^{k,l} = |\delta^2_{i-1,j,k,l}|$, put $(\delta^2_{i-1,j,k,l})^c = 0,$ and
          $(\delta^1_{i,j,k,l})^c = \delta^1_{i,j,k,l} - \delta^2_{i-1,j,k,l}$
          $(\delta^3_{i,j-1,k,l})^c = \delta^3_{i,j-1,k,l} - \delta^2_{i-1,j,k,l}$
          $(\delta^4_{i-1,j-1,k,l})^c = \delta^4_{i-1,j-1,k,l} - \delta^2_{i-1,j,k,l}$
        iii) $\Gamma_{i,j}^{k,l} = |\delta^3_{i,j-1,k,l}|$, put $(\delta^3_{i,j-1,k,l})^c = 0,$ and
          $(\delta^1_{i,j,k,l})^c = \delta^1_{i,j,k,l} - \delta^3_{i,j-1,k,l}$
          $(\delta^2_{i-1,j,k,l})^c = \delta^2_{i-1,j,k,l} - \delta^3_{i,j-1,k,l}$
          $(\delta^4_{i-1,j-1,k,l})^c = \delta^4_{i-1,j-1,k,l} - \delta^3_{i,j-1,k,l}$
        iv) $\Gamma_{i,j}^{k,l} = |\delta^4_{i-1,j-1,k,l}|$, put $(\delta^4_{i-1,j-1,k,l})^c = 0,$, and
          $(\delta^1_{i,j,k,l})^c = \delta^1_{i,j,k,l} - \delta^4_{i-1,j-1,k,l}$
          $(\delta^2_{i-1,j,k,l})^c = \delta^2_{i-1,j,k,l} - \delta^4_{i-1,j-1,k,l}$
          $(\delta^3_{i,j-1,k,l})^c = \delta^3_{i,j-1,k,l} - \delta^4_{i-1,j-1,k,l}$
      second case : $\Gamma_{i,j}^{k,l} > \epsilon_{k,l}$, put
          $(\delta^1_{i,j,k,l})^c = \delta^1_{i,j,k,l}$
          $(\delta^2_{i-1,j,k,l)})^c = \delta^2_{i-1,j,k,l}$
          $(\delta^3_{i,j-1,k,l})^c = \delta^3_{i,j-1,k,l}$
          $(\delta^4_{i-1,j-1,k,l})^c = \delta^4_{i-1,j-1,k,l}$
  End(j)
End(i).

In order to give an error bound between $f_{k,l}$ and $f^c_{k,l}$, we need the following lemma.

Lemma 4.1.

$$\left|\delta^{k,s}_{i,j} - (\delta^{k,s}_{i,j})^c\right| \leq \epsilon_{k,l} \quad \textit{and} \quad \left|\overline{\delta}^{k,s}_{i,j} - (\overline{\delta}^{k,s}_{i,j})^c\right| \leq \epsilon_{k,l}$$

$$\left|\delta^{r,l}_{i,j} - (\delta^{r,l}_{i,j})^c\right| \leq \epsilon_{k,l} \quad \textit{and} \quad \left|\overline{\delta}^{l,r}_{i,j} - (\overline{\delta}^{r,l}_{i,j})^c\right| \leq \epsilon_{k,l}$$

$$\textit{and} \quad \left|\delta^t_{i,j,k,l} - (\delta^t_{i,j,k,l})^c\right| \leq \epsilon_{k,l} \quad \textit{for} \quad t = 1, 2, 3, 4.$$

Proof. From Algorithm 4.1, we easily verify that

$$\delta_{i,j}^{k,s} - (\delta_{i,j}^{k,s})^c = \delta_{i,j}^{k,s} \ \text{ if } \left|\delta_{i,j}^{k,s}\right| \leq \epsilon_{k,l} \ \text{ and } \ \delta_{i,j}^{k,s} - (\delta_{i,j}^{k,s})^c = 0 \ \text{ if } \left|\delta_{i,j}^{k,s}\right| > \epsilon_{k,l},$$

then, we have $\left|\delta_{i,j}^{k,s} - (\delta_{i,j}^{k,s})^c\right| \leq \epsilon_{k,l}$. A similar technique can be used for showing the other inequalities of the Lemma. ∎

Theorem 4.2. *The compressed interpolant $f_{k,l}^c$ given by Algorithm 4.1 is of class $\mathcal{C}^{k,l}$ on Ω. Moreover, we have the following estimations of the error*

$$\left\|g_{k-1,l-1}^1 - (g_{k-1,l-1}^1)^c\right\|_\infty \leq \epsilon_{k,l}\frac{h^k}{k!4^k}\sigma_{l-1},$$

$$\left\|g_{k-1,l-1}^2 - (g_{k-1,l-1}^2)^c\right\|_\infty \leq \epsilon_{k,l}\frac{\hbar^l}{l!4^l}\sigma_{k-1},$$

$$\left\|g_{k-1,l-1}^3 - (g_{k-1,l-1}^3)^c\right\|_\infty \leq \epsilon_{k,l}\frac{h^k\hbar^l}{k!l!4^{k+l}}.$$

Proof. According to Algorithm 4.1, a Hermite data $D^{(r,s)}f(x_i, y_j)$ is removed and substituted for a linear combination of $D^{(r,s)}f_{r-u,s-v}(x_i^\pm, y_j^\pm)$, $0 \leq u, v \leq 1$ and $(u, v) \neq (0, 0)$, when one of the coefficients is smaller than the threshold value $\epsilon_{k,l}$. Then, the modified Hermite data lead to the construction of the compressed interpolant $f_{k,l}^c$ which is obviously of class $\mathcal{C}^{k,l}$ on Ω. On the other hand, using Lemma 4.1, we obtain

$$\left|g_{k-1,l-1}^1(x,y) - (g_{k-1,l-1}^1)^c(x,y)\right| \leq \epsilon_{k,l}\frac{h^k}{k!4^k}\sigma_{l-1} \quad \text{for all } x \in [x_i, x_{i+1}],\ 0 \leq i \leq n-1.$$

Thus, $\left\|g_{k-1,l-1}^1 - (g_{k-1,l-1}^1)^c\right\|_\infty \leq \epsilon_{k,l}\frac{h^k}{k!4^k}\sigma_{l-1}$.

Using a similar technique, one can show the other inequalities of the Theorem. ∎

According to Theorem 4.2, it is clear that when h and $\hbar$ are very small, the function $f_{k,l}^c$ is very close to $f_{k,l}$. This fact is illustrated in the following example.

4.1. Numerical example

Using Algorithm 4.1, we will compress the interpolant $f_{2,2}$ of the function $f(x, y) = \cos\left(\frac{x^2-y}{2}\right)$ studied in the preceding example. Recall that the number of coefficients used for the construction of $f_{2,2}$ is equal to 368. The following table shows the numbers of coefficients removed after thersholding with different values of $\epsilon = (\epsilon_1, \epsilon_2)$ using Algorithm 4.1.

$\epsilon_2 \diagdown \epsilon_1$	0.01	0.02	0.03	0.08	0.15	0.2	0.5	0.7
0.01	49	52	56	72	93	95	104	108
0.04	59	62	66	82	103	105	114	118
0.1	61	64	68	84	105	107	116	120
0.3	72	75	79	95	116	118	127	131

The left graph in Figure 2 stands for the compressed Hermite interpolant $f^c_{2,2}$ obtained from Algorithm 4.1 by using the threshold value $\epsilon = (0.01, 0.01)$. The right graph corresponds to the error function $(f_{2,2} - f^c_{2,2})$.

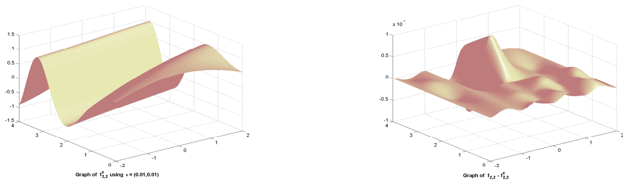

Figure 2.

5. Conclusion

Starting from the results given in [4] and [6], it was proposed in [7] a method which allows to compute recursively the tensor product Hermite interpolant $f_{k,l}$. This method is similar to the one studied in this paper, but the computational cost for evaluating $f_{k,l}$ with the latter is better. Indeed, as given in [7], $f_{k,l}$ can be written as a sum of $f_{k-1,l-1}$ and $l + k + 1$ detail functions but, according to Theorem 3.1, we need here only 3 detail functions. Moreover, it is easy to verify that the number of coefficients needed for computing the detail functions in the decomposition of $f_{k,l}$ using the method introduced in [7] is $4nm(k + l + 1)$, while we need only $2n(m + 1)l + 2m(n + 1)k + 4nm$ coefficients if we use the method developed here. Numerical experiments have still to be done in order to give further information concerning these two methods.

Obviously, the above decomposition of $f_{k,l}$ has several advantages, it allows to compute this interpolant step by step using basis functions which have simple expressions. But this method is adapted only to tensor product Hermite interpolants. Hence, it is interesting to study the decomposition of any bivariate Hermite interpolant. With regard to this topic, we have proposed in [8] a method allowing to build recursively bivariate Hermite polynomial interpolants defined on triangles, and in [5] a hierarchical computation of particular Hermite spline interpolants of class $\mathcal{C}^k$ on $\mathbb{R}^2$. The development of other applications of these results is still under investigation.

6. References

[1] DAHMEN W., OSWALD P., SHI X.-Q., "C^1-hierarchical bases", *J. Comput. Appl. Math*, vol. 51, 1994, 37-56.

[2] de Boor C., *A practical guide to splines*, Springer Verlag, Berlin (1978).

[3] Faber G., "Über stetige Funktionen", *Math. Ann*, vol. 66, 1909, 81-94.

[4] MAZROUI A., SBIBIH D., TIJINI.A, "A recursive construction of Hermite interpolants and applications", to appear in *JCAM*.

[5] MAZROUI A., SBIBIH D., TIJINI.A, "Recursive Computation of Bivariate Hermite Spline Interpolants", submitted.

[6] MAZROUI A., SBIBIH D., TIJINI.A, "A simple method for smoothing function and compressing Hermite data", *Adv. Comput. Math*, vol. 23, 2005, 279-297.

[7] MAZROUI A., SBIBIH D., TIJINI.A, "A recursive method for construction of tensor product Hermite interpolants", *Curve and Surface Design, Saint-Malo 2002, T.Lyche, M.-L.Mazure, and L.L Shcumaker, (eds), Nashboro Press, Brenetwood*, 2003, pp. 303-314.

[8] MAZROUI A., SBIBIH D., TIJINI.A, "Hierarchical computation of bivariate Hermite interpolants", *Curve and Surface Design, Saint-Malo 2002, T.Lyche, M.-L.Mazure, and L.L Shcumaker, (eds), Nashboro Press, Brenewood*, 2003, pp. 315-324.

[9] MAZROUI A., MRAOUI H., SBIBIH D., TIJINI.A, "A new method for smoothing surfaces and computing Hermite interpolants", Submitted.

[10] OSWALD P., "L_p approximation durch Reihen nach dem Haar-Orthogonal-System und dem Faber-Schauder-System", *J. Approx. Theory*, vol. 33, 1981, 1-27.

[11] OSWALD P., *Multilevel Finite Approximation: Theory and Applications,* Teubner, Stuttgart, 1988.

[12] OSWALD P., "Hierarchical conforming finite element methods for the biharmonic equation", *SIAM J. Numer. Anal*, vol. 29, 1992, 1610-1625.

[13] RIORDAN J., *Combinatorial identities*, Wiley Series in Probability and Mathematical Statistics, 1989.

[14] YSERENTANT H., "On the multilevel splitting of finite element spaces", *Numer. Math*, vol. 49, 1986, 379-412.