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Singular perturbations on the infinite time interval

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ABSTRACT. We consider the slow and fast systems that belong to a small neighborhood of an unperturbed problem. We study the general case where the slow equation has a compact positively invariant subset which is asymptotically stable, and meanwhile the fast equation has asymptotically stable equilibria (Tykhonov's theory) or asymptotically stable periodic orbits (Pontryagin–Rodygin's theory). The description of the solutions is by this way given on infinite time interval. We investigate the stability problems derived from this results by introducing the notion of practical asymptotic stability. We show that some particular subsets of the phase space of the singularly perturbed systems behave like asymptotically stable sets. Our results are formulated in classical mathematics. They are proved within Internal Set Theory which is an axiomatic approach to Nonstandard Analysis.

RÉSUMÉ. On considère les systèmes lents-rapides appartenant à un petit voisinage d'un problème non perturbé. On étudie le cas général où l'équation lente admet un sous-ensemble compact positivement invariant qui soit asymptotiquement stable tandis que l'équation rapide a des équilibres asymptotiquement stables (théorie de Tykhonov) ou des cycles limites stables (théorie de Pontryagin). La description des solutions est de ce fait donnée sur des intervalles de temps infinis. On examine les problèmes de stabilité découlant de ces résultats en introduisant la notion de stabilité pratique. On montre que certains sous-ensembles de l'espace de phases des systèmes singulièrement perturbés se comportent comme des ensembles asymptotiquement stables. Les résultats sont formulés classiquement mais sont démontrés dans le cadre de la théorie IST, une approche axiomatique de l'Analyse Non Standard.

KEYWORDS : invariant sets, practical asymptotic stability, singular perturbations, nonstandard analysis

MOTS-CLÉS : ensembles invariants, stabilité asymptotique pratique, perturbations singulières, analyse non standard

1. Introduction

Let us provide the set

$$\mathcal{T} = \{(\Omega, f, g, \alpha, \beta) : \Omega \text{ open subset of } \mathbb{R}^{n+m}, (\alpha, \beta) \in \Omega,$$
$$f : \Omega \to \mathbb{R}^n,\ g : \Omega \to \mathbb{R}^m \text{ continuous}\}$$

with the topology defined as follows : a neighborhood system of an element

$$(\Omega_0, f_0, g_0, \alpha_0, \beta_0) \in \mathcal{T}$$

is generated by the sets

$$V(D, a) = \{(\Omega, f, g, \alpha, \beta) \in \mathcal{T} :\ D \subset \Omega,\ ||f - f_0||_D < a,\ ||g - g_0||_D < a,$$
$$||\alpha - \alpha_0|| < a,\ ||\beta - \beta_0|| < a\},$$

where D is a compact subset of Ω and a a positive real number. Here,

$$||h||_D = \sup_{u \in D} ||h(u)||,$$

where h is defined on D with values in a normed space. Let us call it "the topology of uniform convergence on compacta" and consider the initial value problem

$$\begin{aligned} \varepsilon \dot{x} = f(x, y),\ \ x(0) = \alpha, \\ \dot{y} = g(x, y),\ \ y(0) = \beta, \end{aligned} \tag{1}$$

corresponding to an element $(\Omega, f, g, \alpha, \beta)$ of $\mathcal{T}$ where ε is a small positive real number and $(\dot{\cdot}) = d/dt$. The variable x is called a *fast variable*, y is called a *slow variable*. We propose to study the system (1) with ε small enough and $(\Omega, f, g, \alpha, \beta)$ sufficiently close to an element $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ of $\mathcal{T}$ in the sense of the defined topology. It is a singular perturbation problem because the multiplication of the derivative by ε fails with the use of theory of continuous dependence of the solutions with respect to the parameters and the initial conditions. Problems as (1) gain their special structure from the presence of two time scales. They are called *slow and fast systems*. We define the *fast equation* by

$$x' = f_0(x, y),\ y\ parameter, \tag{2}$$

where $(') = d/d\tau$ and $\tau = t/\varepsilon$.

Suppose the fast equation admits, for all y in a compact subset Y of $\mathbb{R}^m$ an asymptotically stable equilibrium point $x = \xi(y)$, uniformly in the parameter $y \in Y$, ξ being a continuous function on Y. We define the *slow equation of* (1) by

$$\dot{y} = g_0(\xi(y), y),$$

and the *reduced problem* by

$$\dot{y} = g_0(\xi(y), y),\ y(0) = \beta_0. \tag{3}$$

Problem (3) is supposed to have a unique solution $\bar{y}(t)$ on an interval $[0, T]$. It is mainly proved in [13] (see Theorem 4.16 in this paper) that, for ε small enough and $(\Omega, f, g, \alpha, \beta)$

close to $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$, every solution $(x(t), y(t))$ of (1) is defined at least on $[0, T]$ and is approximated by $(\xi(\bar{y}(t)), \bar{y}(t))$ for all $t \in]0, T]$. This approximation can be extended to $t = 0$ only for $y(t)$. Indeed, a boundary layer phenomena is observed for $x(t)$ at $t = 0$. Actually, the fast variable is approximated by the solution of the so-called *boundary layer equation*

$$x' = f_0(x, \beta_0),\ x(0) = \alpha_0. \tag{4}$$

Now, suppose that the fast equation rather admits for all y in a compact subset G of $\mathbb{R}^m$ a $T(y)$-periodic non-trivial solution $x^*(\tau, y)$. If the corresponding periodic orbit Γ_y is asymptotically stable, uniformly in $y \in G$, we consider the following averaged system as the reduced problem :

$$\dot{y} = \frac{1}{T(y)} \int_0^{T(y)} g_0(x^*(\tau, y), y)\, d\tau,\ y(0) = \beta_0. \tag{5}$$

If $\bar{y}(t)$ is the unique solution of (5), defined on $[0, T]$, it is mainly proved in [20] (see Theorem 4.17 in the present paper) that for ε small enough and $(\Omega, f, g, \alpha, \beta)$ close to $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$, every solution $(x(t), y(t))$ of (1) is defined at least on $[0, T]$. The component $y(t)$ is close to $\bar{y}(t)$ for $0 \leq t \leq T$ and the component $x(t)$ is close to the orbit $\Gamma_{\bar{y}(t)}$ for $0 < t \leq T$. There is a boundary layer at $t = 0$. The boundary layer equation (4) describes the fast variable. Hence, after a fast transition, the trajectory rolls up quickly around the manifold generated by the cycles with a slow drift of the y-component.

For smooth vector fields, when the reduced problem has a limit cycle Γ, the question of wether the singularly perturbed problem has a periodic orbit Γ_ε near Γ or not for small values of ε has been studied by authors like K. O. Friedrichs and W. Wasow [7], L. Flatto and N. Levinson [6], N. D. V. Anosov [1], N. Fenichel [5] and recently F. Verhulst et al. [23, 24]. In [1, 6], it was established that the orbit Γ of period $\mathcal{P}$ can be continued to a family Γ_ε of closed orbits if(i) Γ, as an orbit of the reduced problem, has 1 as a simple Floquet multiplier, (ii) for all $y \in \Gamma$, the equilibrium point $x = \xi(y)$ of the boundary layer equation is hyperbolic. This kind of results need strong conditions which can ever ensures both the uniqueness of Γ_ε for ε small enough with a period tending to $\mathcal{P}$ as $\varepsilon \to 0$ and the property of asymptotic phase.

Our aim is first to approximate the solutions of (1) on infinite time interval. More exactly, we will suppose that the slow equation has an asymptotically stable compact subset $\mathcal{M}$ which is positively invariant. The approximations on unbounded time interval are then given in Section 2 (Theorem 2.3 and Theorem 2.4). If the subset $\mathcal{M}$ is reduced to an equilibrium point, Theorems 2.3 and 2.4 are nothing else than the extension for all $t \geq 0$ of Theorems 4.16 and 4.17 already established in [13] and [20] : the solutions of the singularly perturbed system not only live all the time, but they also stay close to the curve formed by the concatenation of the trajectories of the fast and the slow equations. In Section 3, some examples show that there is no hope to obtain stability results with our weak conditions. Nevertheless, the theorems of the preceding section lead to interesting statements in terms of practical stability : when the reduced problem has an asymptotically stable positively invariant compact subset, it arises a subset of the whole phase space which seems to be asymptotically stable when $\varepsilon \to 0$ for the singularly perturbed problem (see Theorems 3.2 and 3.3). In the present work, the results are formulated in classical mathematics and proved within *Internal Set Theory* (IST) [15] which is an axiomatic approach of *Nonstandard Analysis* (NSA) [17]. We characterize notions of stability, practical stability and perturbations and translate our main results in nonstandard words in Section 4. Actually, we rather introduce the notion of s-stability which will lead to results generalizing those of Theorems 3.2 and 3.3. We prove our results in Section 5.

2. Approximation results

2.1. Some definitions

We recall the notion of asymptotic stability (in Lyapunov sense) (see [4, 8] for example) by considering a differential system

$$\dot{x} = f(x), \tag{6}$$

where f is continuous on an open subset U of $\mathbb{R}^n$.

Definition 2.1 *1. A subset $\mathcal{M}$ of U is said to be **positively invariant** for the system (6) if every solution such that $x(0) \in \mathcal{M}$ is defined for all $t > 0$ and satisfies $x(t) \in \mathcal{M}$.*

*2. A bounded subset $\mathcal{M}$ of U is said to be **stable** for the system (6) if, for each $\mu > 0$, there exists $\eta > 0$ such that any solution $x(t)$ of (6) for which $\operatorname{dis}(x(0), \mathcal{M}) < \eta$ can be continued for all $t \geq 0$ and satisfies $\operatorname{dis}(x(t), \mathcal{M}) < \mu$, where $\operatorname{dis}(x(t), \mathcal{M})$ denotes the distance from the point $x(t)$ to the set $\mathcal{M}$ given by $\inf_{m \in \mathcal{M}} ||x(t) - m||$.*

*3. $\mathcal{M}$ is said to be **attractive** if it admits a neighborhood $\mathcal{V}$ (basin of attraction) such that any solution $x(t)$ of (6) for which $x(0) \in \mathcal{V}$ can be continued for all $t \geq 0$ and satisfies $\lim_{t \to \infty} \operatorname{dis}(x(t), \mathcal{M}) = 0$.*

*3. $\mathcal{M}$ is **asymptotically stable** if it is stable and attractive.*

Consider the case where the system

$$\dot{x} = f(x, y), \tag{7}$$

depends on a parameter y which belongs to a set Y. Let $\mathcal{M}_y$ be an asymptotically stable bounded subset of U for each value of y in Y.

Definition 2.2 *The basin of attraction of the asymptotically stable bounded subset $\mathcal{M}_y$ of U is said to be **uniform over** Y for (7) if there exists a real number $a > 0$ such that, for all y in Y, the set $\{x \in \mathbb{R}^n \colon \operatorname{dis}(x, \mathcal{M}_y) \leq a\}$ is in the basin of attraction of $\mathcal{M}_y$.*

2.2. The main theorems

Let us first make the following assumptions :

$T1$: *The fast equation (2) has the property of uniqueness of the solutions with the prescribed initial conditions.*

The *slow manifold* of the system (1) is defined by the set of the points of $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$f_0(x, y) = 0. \tag{8}$$

It is namely the set of equilibria of the fast equation (2).

$T2$: *There exists a continuous mapping $\xi : Y \to \mathbb{R}^n$, Y being a compact subset of $\mathbb{R}^m$ with a non empty interior, such that $(\xi(y), y) \in \Omega_0$ for all $y \in Y$ and $f_0(\xi(y), y) = 0$. Moreover, for all $y \in Y$, $x = \xi(y)$ is an isolated root of the equation (8), that is there exists a real number $\delta > 0$ such that if $y \in Y$, $\|x - \xi(y)\| < \delta$ and $x \neq \xi(y)$ then $f_0(x, y) \neq 0$.*

The subset $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x = \xi(y),\ y \in Y\}$ of $\mathbb{R}^n \times \mathbb{R}^m$ is an m-dimensional manifold included in the slow manifold.

$T3$: *For all y in Y, the equilibrium point $x = \xi(y)$ of the fast equation (2) is asymptotically stable and its basin of attraction is uniform over Y.*

By substituting $\xi(y)$ to x in the second equation of (1) we obtain the slow equation

$$\dot{y} = g_0(\xi(y), y), \tag{9}$$

which will be defined in the interior $\mathring{Y}$ of the compact set Y.

$T4$: *The slow equation (9) has the property of uniqueness of the solutions with the prescribed initial conditions.*

$T5$: *The point β_0 is in $\mathring{Y}$ and α_0 is in the basin of attraction of the equilibrium point $x = \xi(\beta_0)$.*

Theorem 2.3 *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be an element of $\mathcal{T}$ and $\xi : Y \to \mathbb{R}^n$ be a continuous function. Let hypotheses $T1$ to $T5$ be satisfied. Let $\mathcal{M}$ be a closed subset in $\mathring{Y}$ which is positively invariant for the slow equation (9). Suppose that $\mathcal{M}$ is asymptotically stable for (9) with β_0 in its basin of attraction. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of the boundary layer equation (4) and of the reduced problem (3). Then, for all $\eta > 0$, there exists a real number $\varepsilon^* > 0$ and a neighborhood $\mathcal{V}$ of the element $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ of $\mathcal{T}$ such that, for all $\varepsilon < \varepsilon^*$ and all $(\Omega, f, g, \alpha, \beta) \in \mathcal{V}$, any solution $(x(t), y(t))$ of the problem (1) is defined for all $t \geq 0$ and there exist $\omega > 0$ and $\omega' > 0$ such that :*

$$\begin{array}{l} \varepsilon\omega < \eta,\ 1/\omega' < \eta, \\ \|x(\varepsilon\tau) - \tilde{x}(\tau)\| < \eta \text{ for } 0 \leq \tau \leq \omega, \\ \|x(t) - \xi(\bar{y}(t))\| < \eta \text{ for } t \in [\varepsilon\omega, \omega'] \\ \|y(t) - \bar{y}(t)\| < \eta \text{ for } t \in [0, \omega'], \\ \mathrm{dis}(y(t), \mathcal{M}) < \eta \text{ for } t \geq \omega', \\ \mathrm{dis}(x(t), \xi(\mathcal{M})) < \eta \text{ for } t \geq \omega'. \end{array} \tag{10}$$

The set $\xi(\mathcal{M})$ is the range of $\mathcal{M}$ by the function ξ. Note that $\mathcal{M}$ is necessarily a compact subset of $\mathring{Y}$ as will be all the positively invariant subsets of the following. When the fast equation has limit cycles, the preceding theorem does not fit. Let us make instead the following assumptions, the first one being nothing else than $(T1)$.

$P1$: *The fast equation (2) has the property of uniqueness of the solutions with the prescribed initial conditions.*

$P2$: *There exists a family of solutions $x^*(\tau, y)$ depending continuously on $y \in G$, where G is a compact subset of $\mathbb{R}^m$ with a non empty interior, such that $x^*(\tau, y)$ is a periodic solution of the fast equation (2) of period $T(y) > 0$, the mapping $y \to T(y)$ is continuous.*

$P3$: *The closed orbit Γ_y corresponding to the periodic solution $x^*(\tau, y)$ is asymptotically stable and its basin of attraction is uniform over G.*

What precede means that the cycle[1] Γ_y depends continuously on y and is locally unique, that is, there exists an neighborhood W of Γ_y such that the equation (2) has no other cycle in W.

We define the slow equation in the interior $\mathring{G}$ of G by the averaged system

$$\dot{y} = \bar{g}_0(y) := \frac{1}{T(y)} \int_0^{T(y)} g_0(x^*(\tau, y), y) d\tau. \tag{11}$$

Assume what follows :

$P4$: *The slow equation (11) has the uniqueness of the solutions with the prescribed initial conditions.*

$P5$: *The point β_0 is in $\mathring{G}$ and α_0 is in the basin of attraction of Γ_{β_0}.*

Theorem 2.4 *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be an element of $\mathcal{T}$. Let hypotheses $P1$ to $P5$ be satisfied. Let $\mathcal{M}$ be a closed subset in $\mathring{G}$ which is positively invariant for the slow equation (11). Suppose that $\mathcal{M}$ is asymptotically stable for (11) with β_0 in its basin of attraction. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of the boundary layer equation (4) and of the reduced problem (5). Then, for all $\eta > 0$, there exists a real number $\varepsilon^* > 0$ and a neighborhood $\mathcal{V}$ of the element $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ of $\mathcal{T}$ such that, for all $\varepsilon < \varepsilon^*$ and all $(\Omega, f, g, \alpha, \beta) \in \mathcal{V}$, any solution $(x(t), y(t))$ of the problem (1) is defined for all $t \geq 0$ and there exist $\omega > 0$ and $\omega' > 0$ such that :*

$$\begin{aligned}
&\varepsilon\omega < \eta,\ 1/\omega' < \eta, \\
&||x(\varepsilon\tau) - \tilde{x}(\tau)|| < \eta \text{ for } 0 \leq \tau \leq \omega, \\
&||y(t) - \bar{y}(t)|| < \eta \text{ for } t \in [0, \omega'], \\
&\operatorname{dis}(x(t), \Gamma_{\bar{y}(t)}) < \eta \text{ for } t \in [\varepsilon\omega, \omega'], \\
&\operatorname{dis}(y(t), \mathcal{M}) < \eta \text{ for } t \geq \omega', \\
&\operatorname{dis}(x(t), \Gamma_{y(t)}) < \eta \text{ for } t \geq \omega'.
\end{aligned}$$

The proofs of Theorems 2.3 and 2.4 are postponed to Section 4.5. The particular case of Theorem 2.3, when $\mathcal{M}$ is an asymptotically stable equilibrium point, is Theorem 2 of [13]. The result therein concerns the case where the slow equation (9) has an asymptotically stable equilibrium point y_∞ in $\mathring{Y}$ with β_0 in its basin of attraction. For ε small enough and for $(\Omega, f, g, \alpha, \beta)$ close to $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$, under the assumptions $T1$ to $T5$, any solution $(x(t), y(t))$ of (1) is close to $(\xi(\bar{y}(t), \bar{y}(t))$ for all $t > 0$. Here, one takes $\mathcal{M} = \{y_\infty\}$. The particular case of Theorem 2.4, when $\mathcal{M}$ is an asymptotically stable equilibrium point, is Theorem 2.3 of [20]. Under the assumptions $P1$ to $P5$, any solution $(x(t), y(t))$ of (1) is close to $(\Gamma_{\bar{y}(t)}, \bar{y}(t))$ for all $t > 0$. Finally, one can mention the important particular cases where the slow equations (9) and (11) admit an asymptotically stable cycle.

1. Actually, the term cycle should be used for planar systems. A stable limit cycle is the asymptotically stable closed orbit associated to an orbitally asymptotically stable periodic solution.

3. Stability results

3.1. Some examples

Suppose that the slow equation (9) admits, say an asymptotically stable equilibrium point y_∞ and that the corresponding point $(\xi(y_\infty), y_\infty)$ lying on the slow manifold is an equilibrium point of (1). To deduce asymptotic stability of $(\xi(y_\infty), y_\infty)$ for the whole system (1) for ε small enough, Tykhonov theory needs strong assumptions like smoothness of the field and exponential stability of the equilibrium points of both slow and fast equations (see [9], Theorem 9.3, page 380 [2])). It is also the case in Pontryagin-Rodygin theory. In most of the applications, it is quite reasonable to impose conditions of exponential or uniform exponential stability instead of the simple asymptotic stability (asymptotic stability which is not exponential is said to be critical). The following examples show the non robustness of the critical asymptotic stability. Before, one has to keep in mind that the condition of exponential stability for such results is sufficient, but not necessary as shown in the following example (see also [9], Theorem 9.2, page 377).

Example 1 ([9], Example 9.9, page 377) The origin of the system

$$\begin{aligned} \varepsilon\dot{x} &= -x + \varepsilon(x - y^3), \\ \dot{y} &= -y^3 + x. \end{aligned} \tag{12}$$

is an equilibrium point. The slow equation is give by

$$\dot{y} = -y^3.$$

Its origin is asymptotically but not exponentially stable. The fast equation is

$$x' = -x,$$

and its origin is exponentially stable. It is shown in [9] that for ε small enough, the origin of the the whole system (12) is asymptotically stable by the use of the so-called quadratic Lyapounov functions. It is well explained in the same reference that the existence of quadratic Lyapounov functions do not always imply exponential stability which is a particular case.

Example 2 The fast equation of the planar slow and fast system

$$\begin{aligned} \varepsilon\dot{x} &= -x + y, \\ \dot{y} &= -y^2 x + \varepsilon y, \end{aligned} \tag{13}$$

where $\varepsilon > 0$ is a small real parameter, is given by

$$x' = -x + y.$$

Assumption $T1$ is clearly satisfied for the fast equation. The slow curve is the line $x = \xi(y) := y$ (Assumption $T2$). All it points are (globally) asymptotically stable equilibria

2. Actually, the theorem therein concerns the nonautonomous case but can be stated or the autonomous one.

of the fast equation for each value of y (Assumption $T3$). By substituting $\xi(y)$ to x and 0 to ε in the second equation of the system, we obtain the slow equation

$$\dot{y} = -y^3,$$

which satisfies Assumption $T4$. The origin $y = 0$ of the slow equation is (globally) asymptotically stable (Assumption $T6$). One can apply the preceding theory (more exactly the extension of Tykhonov's Theorem for infinite time interval) for any initial condition $(x_0, y_0) \neq (0,0)$ (Assumption $T5$) to have

$$\begin{aligned} &\forall t \geq 0,\ \lim_{\varepsilon \to 0} y(t,\varepsilon) = \bar{y}(t), \\ &\forall t > 0,\ \lim_{\varepsilon \to 0} x(t,\varepsilon) = \xi(\bar{y}(t)) = \bar{y}(t), \end{aligned}$$

where $(x(t,\varepsilon), y(t,\varepsilon))$ is the solution of the whole system with initial condition (x_0, y_0) and $\bar{y}(t)$ is the solution of the reduced problem. Now, knowing that $\bar{y}(t)$ goes to 0 when t goes towards $+\infty$, we obtain that

$$\lim_{\varepsilon \to 0,\ t \to +\infty} (x(t,\varepsilon), y(t,\varepsilon)) = (0,0).$$

However, this limit does not mean that the origin of the whole system, which is an equilibrium point, is asymptotically stable, not even attractive. It is easy to check, by linearization, that it is a saddle point for any positive value of ε.

Example 3 The following example

$$\begin{aligned} \varepsilon \dot{x} &= -x^3 + \varepsilon x, \\ \dot{y} &= -y + x^2, \end{aligned} \tag{14}$$

is given just to exhibit a case where the origin of the fast equation $x' = -x^3$ is critically asymptotically stable whereas the origin of the slow equation $\dot{y} = -y$ is exponentially stable. Onee can claim that any solution of the singularly perturbed system goes towards the equilibrium point $(0,0)$ when $\varepsilon \to 0$ and $t \to +\infty$. Nevertheless, the origin $(0,0)$ is a saddle point for each value of $\varepsilon > 0$.

Example 4 The fast and the slow equation of the system

$$\begin{aligned} \varepsilon \dot{x} &= -x + \varepsilon, \\ \dot{y} &= -y + \varepsilon x, \end{aligned}$$

are respectively given by $x' = -x$ and $\dot{y} = -y$. In both cases, the origin is an exponentially stable equilibrium point. Any solution of the singularly perturbed system goes towards the equilibrium point $(0,0)$ when $\varepsilon \to 0$ and $t \to +\infty$. Nevertheless, the origin $(0,0)$ is not even an equilibrium point of the whole system.

Example 5 The following tridimensional singularly perturbed system (Σ)

$$\begin{aligned} \varepsilon \dot{x}_1 &= -x_2 + x_1 \left(\sqrt{x_1^2 + x_2^2} - 1 + \varepsilon\right)\left(\sqrt{x_1^2 + x_2^2} - 1 - \varepsilon\right)(1 - x_1^2 - x_2^2), \\ \varepsilon \dot{x}_2 &= x_1 + x_2 \left(\sqrt{x_1^2 + x_2^2} - 1 + \varepsilon\right)\left(\sqrt{x_1^2 + x_2^2} - 1 - \varepsilon\right)(1 - x_1^2 - x_2^2), \\ \dot{y} &= -y^3 x_1^2, \end{aligned}$$

is written under cylindrical coordinates $(x_1 = r\cos\theta, x_2 = r\sin\theta, y)$ as

$$\begin{aligned}
&\varepsilon\dot{r} = r(r-1+\varepsilon)(r-1-\varepsilon)(1-r^2),\\
&\varepsilon\dot{\theta} = 1,\\
&\dot{y} = -r^2y^3\cos^2\theta.
\end{aligned}$$

The equilibrium point $r = 1$ of the first equation of

$$\begin{aligned}
&r' = r(r-1)^2(1-r^2),\\
&\theta' = 1,
\end{aligned} \tag{15}$$

corresponds to a stable limit cycle Γ_y the basin of attraction of which is the whole plane (x_1, x_2) except the origin. So, it is uniform in y (actually the cycle does not depend on y). A 2π-periodic solution associated to the cycle is $x^*(\tau, y) = (\cos\tau, \sin\tau)$. According to Pontryagin-Rodygin's theory, the equation describing the slow motion is given by the average system

$$\dot{y} = -\frac{1}{2\pi}\int_0^{2\pi} y^3\cos^2\tau d\tau = -\frac{y^3}{2}, \tag{16}$$

the origin $y = 0$ of which is an asymptotically stable equilibrium. The preceding theory, and more exactly the extension of Pontryagin-Rodygin's Theorem in [20], implies that for any initial condition (x_1^0, x_2^0, y^0) of (Σ), apart from the origin, we have

$$\forall t \geq 0,\ \lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}(t), \quad \forall t > 0,\ \lim_{\varepsilon\to 0} \mathrm{dis}((x_1(t,\varepsilon), x_2(t,\varepsilon)), \Gamma_{\bar{y}(t)}) = 0.$$

Here, $(x_1(t,\varepsilon), x_2(t,\varepsilon), y(t,\varepsilon))$ is the solution of the singularly perturbed problem (Σ) with initial condition (x_1^0, x_2^0, y^0) and $\bar{y}(t)$ is the solution of (16) starting from y^0. Since $\bar{y}(t) \to 0$ when $t \to +\infty$, we have the following limit

$$\lim_{\varepsilon\to 0,\, t\to+\infty} \mathrm{dis}((x_1(t,\varepsilon), x_2(t,\varepsilon), y(t)), \Gamma_0 \times \{0\}) = 0.$$

This does not either mean that the closed curve $\Gamma_0 \times \{0\}$ in the phase space $\mathbb{R}^3$ is an asymptotically stable orbit. One can see that for each $\varepsilon > 0$, the cylinder generated by the cycles Γ_y is "repelling" for (Σ), being located between two "attracting" cylinders corresponding to $r = 1-\varepsilon$ and $r = 1+\varepsilon$. In other words, the "exact fast dynamics" of the whole problem, that is the subsystem

$$\begin{aligned}
&r' = r(r-1+\varepsilon)(r-1-\varepsilon)(1-r^2),\\
&\theta' = 1,
\end{aligned}$$

which is a regular perturbation of (15), admits for all y and all $\varepsilon > 0$ two "stable limit cycles" $r = 1-\varepsilon$ and $r = 1+\varepsilon$ surrounding the "unstable limit cycle" $r = 1$.

Example 6 The slow dynamic associated to the following slow-fast system written in cylindrical coordinates $(x = x, y_1 = r\cos\theta, y_2 = r\sin\theta)$

$$\begin{aligned}
&\varepsilon\dot{x} = -x + r\cos\theta,\\
&\dot{r} = r(r-1+\varepsilon)(r-1-\varepsilon)(1-r^2),\\
&\dot{\theta} = 1,
\end{aligned} \tag{17}$$

is approximated by the slow equation

$$\begin{aligned}
&\dot{r} = -r(r-1)^3(r+1),\\
&\dot{\theta} = 1,
\end{aligned}$$

having the unit circle Γ as a unique asymptotically stable cycle. The equilibrium points of the fast equation $x' = -x + y_1$ are all points of the slow surface $x = \xi(y_1, y_2) := y_1$ and they are (globally) asymptotically stable. Let $(x(t,\varepsilon), y_1(t,\varepsilon), y_1(t,\varepsilon))$ be the solution of (17) written in the rectangular coordinates with any initial condition apart the origin. We can deduce from Theorem 2.3 in the special case where $\mathcal{M} = \Gamma$ that the following limits hold

$$\lim_{\varepsilon\to 0,\ t\to+\infty} \operatorname{dis}(x(t,\varepsilon), \xi(\Gamma)) = 0,$$
$$\lim_{\varepsilon\to 0,\ t\to+\infty} \operatorname{dis}((y_1(t,\varepsilon), y_2(t,\varepsilon)), \Gamma) = 0.$$

But even if the subset $\xi(\Gamma) \times \Gamma$ of $\mathbb{R}^3$ was properly a closed orbit of (17), it could not be stable. Indeed, for the subsystem

$$\dot r = r(r - 1 + \varepsilon)(r - 1 - \varepsilon)(1 - r^2),$$
$$\dot\theta = 1,$$

of (17), for any value of $\varepsilon > 0$, the orbit Γ is an unstable limit cycle, surrounded by the two stable limit cycles $r = 1 - \varepsilon$ and $r = 1 + \varepsilon$.

3.2. Practical stability

Although they do not imply the existence of asymptotically stable sets for the singularly perturbed systems, the limits of the preceding examples show a kind of "seeming" stability of certain points or closed curves in the whole phase space. It is known (see [8], Theorem 38.1) that the uniform attractivity, say of the origin of a system, with respect to the initial conditions, implies it asymptotic stability. Moreover, the global asymptotic stability is equivalent to the uniformity of the attractivity for any initial condition in an arbitrarily large ball centered at the origin[3]. The following definition is inspired by lectures given by Lobry and Sari during the CIMPA School-2003 in Tlemcen, *Contrôle Non Linéaire et Applications* [12] (see also [11]). The terminology is yet taken from control theory where it arises in problems of stabilization (see for example [21]). Let us consider the slow and fast system

$$\begin{aligned} \varepsilon\dot x &= f_0(x,y), \\ \dot y &= g_0(x,y), \end{aligned} \tag{18}$$

where $f_0 : \Omega_0 \to \mathbb{R}^n$ and $g_0 : \Omega_0 \to \mathbb{R}^m$ are continuous on an open subsetΩ_0 of $\mathbb{R}^{n+m}$.

Definition 3.1 *A bounded subset $\mathcal{A}$ of Ω_0 is said to be semiglobally practically asymptotically stable (SGPAS) for the system (18) when $\varepsilon \to 0$ if, for every compact neighborhood $\mathcal{K} \subset \Omega_0$ of $\mathcal{A}$ and every neighborhood $\mathcal{O} \subset \Omega_0$ of $\mathcal{A}$, there exist $\varepsilon_0 > 0$ and a real number $T > 0$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, all $t \geq T$ and all $(\alpha, \beta) \in \mathcal{K}$, any solution $(x(t), y(t))$ of (18) with initial condition (α, β) is in $\mathcal{O}$.*

If $\mathcal{K}$ is not arbitrary, the subset $\mathcal{A}$ is said to be practically asymptotically stable (PAS) for the system (18) when $\varepsilon \to 0$.

The definition says that any solution $(x(t), y(t))$ of (18) with initial condition (α, β) satisfies $\lim\ \operatorname{dis}((x(t), y(t)), \mathcal{A}) = 0$ when $t \to +\infty$ and $\varepsilon \to 0$, the convergence being

3. The famous Vinograd example ([**?**], page 191) is a good illustration of attractivity which is not uniform with respect to the initial conditions.

uniform with respect to (α, β) in $\mathcal{K}$ (i.e. $\lim_{t \to +\infty,\ \varepsilon \to 0} Sup_{(\alpha,\beta)\in\mathcal{K}} \mathrm{dis}((x(t), y(t)), \mathcal{A}) = 0$). In other words, for the first part of the definition, any trajectory starting from an arbitrarily point (α, β) which is not close to the boundary of Ω_0 reaches an arbitrarily small neighborhood $\mathcal{O}$ of $\mathcal{A}$ in finite time and for ε small enough. Note that it is required that the finite time T and the threshold ε_0 are the same for any initial condition (α, β).

3.3. The theorems

Finally, practical stability answers to natural questions as the following : "If the slow equation (9) admits, say an asymptotically stable equilibrium point y_∞ or a limit cycle Γ, which part acts, for the whole problem (1), the corresponding point $(\xi(y_\infty), y_\infty)$ or subset $\xi(\Gamma) \times \Gamma$ lying on the slow manifold? They have no reason to be respectively an equilibrium point or a cycle, unless we add strong hypotheses like smoothness and exponential stability. However, for very small values of the parameter ε they look like. Because of the asymptotic feature of this property, an engineer, a biologist,...would be satisfied by a model which seems to tend towards a steady or permanently oscillating state, instead of a perfect stability which is after all ideal. The following results of practical stability deduce from the Theorems 2.3 and 2.4. Their proofs are postponed to Section 4.5.

Theorem 3.2 *Let $f_0 : \Omega_0 \to \mathbb{R}^n$ and $g_0 : \Omega_0 \to \mathbb{R}^m$ be continuous on an open subset Ω_0 of $\mathbb{R}^{n+m}$ satisfying hypotheses $T1$ to $T4$. Let $\mathcal{M}$ be a closed subset in $\mathring{Y}$ which is positively invariant for the slow equation (9). Suppose that $\mathcal{M}$ is asymptotically stable for (9). Then the subset $\xi(\mathcal{M}) \times \mathcal{M}$ of the slow manifold is PAS for the system (18) when $\varepsilon \to 0$.*

Theorem 3.3 *Let $f_0 : \Omega_0 \to \mathbb{R}^n$ and $g_0 : \Omega_0 \to \mathbb{R}^m$ be continuous on an open subset Ω_0 of $\mathbb{R}^{n+m}$ satisfying hypotheses $P1$ to $P4$. Let $\mathcal{M}$ be a closed subset in $\mathring{G}$ which is positively invariant for the slow equation (11). Suppose that $\mathcal{M}$ is asymptotically stable for (11). Then the subset $\bigcup_{y\in\mathcal{M}} (\Gamma_y \times \{y\})$ of Ω_0 is PAS for the system (18) when $\varepsilon \to 0$.*

If the equilibrium points and the positively invariant sets of the fast and slow equations were globally asymptotically stable, we would have results in terms of SGPAS when $\varepsilon \to 0$. A natural consequence of these two theorems is obtained when $\mathcal{M}$ is an asymptotically stable cycle. In the case of Theorem 3.2, the singularly perturbed system (18) seems to have an asymptotically stable cycle on the slow manifold for ε small enough. In the case of Theorem 3.3, we obtain a torus which seems to attract the solutions of (18) when ε is sufficiently small. Hence, in the light of these results, we can come back to the examples above. In Example 2, the equilibria of the fast equation are exponentially stable, the origin of the slow equation is asymptotically stable, the origin is an equilibrium point of the whole system and it is SGPAS when $\varepsilon \to 0$. In Example 3, the origin is asymptotically stable for the fast equation, exponentially stable for the slow equation, it is an equilibrium point of the whole system and this equilibrium is SGPAS when $\varepsilon \to 0$. In Example 4, the origin of both the fast and the slow equation is exponentially stable, but $(0, 0)$ is not an equilibrium point of the singularly perturbed system. However, it is SGPAS when $\varepsilon \to 0$. In Example 5, the fast equation has a unique asymptotically stable cycle Γ_y for each value of y, the origin of the slow equation is asymptotically stable. The subset $\Gamma_0 \times \{0\}$ of $\mathbb{R}^3$ is SGPAS when $\varepsilon \to 0$ (even if it was a cycle, it would not be stable). Finally, in the Example 6, the slow equation has a unique asymptotically stable cycle Γ, the equilibrium