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Analysis of an Age-structured SIL model

with demographics process and vertical transmission

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ABSTRACT. We consider a mathematical SIL model for the spread of a directly transmitted infectious disease in an age-structured population; taking into account the demographic process and the vertical transmission of the disease. First we establish the mathematical well-posedness of the time evolution problem by using the semigroup approach. Next we prove that the basic reproduction ratio R_0 is given as the spectral radius of a positive operator, and an endemic state exist if and only if the basic reproduction ratio R_0 is greater than unity, while the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. We also show that the endemic steady states are forwardly bifurcated from the disease-free steady state when R_0 cross the unity. Finally we examine the conditions for the local stability of the endemic steady states.

RÉSUMÉ. Nous considérons ici un modèle mathématique SIL de transmission directe de la maladie dans une population hôte structurée en âge; prenant en compte les processus démographiques et la transmission verticale de la maladie. Premièrement, nous étudions le caractère bien posé du problème par la théorie des semi-groupes. Ensuite, nous montrons que le taux de reproduction de base R_0 est le rayon spectral d'un opérateur positif; et un équilibre endémique existe si et seulement si R_0 est supérieur à l'unité, tandis que l'équilibre sans maladie est localement asymptotiquement stable si $R_0 < 1$. Nous établissons aussi l'existence d'une bifurcation de l'équilibre sans maladie quand R_0 passe par l'unité. Enfin, nous donnons des conditions nécessaires pour la stabilité locale de l'équilibre endémique.

KEYWORDS : Age-structured model, Semigroup, Basic reproduction ratio, Stability.

MOTS-CLÉS : Modèle structuré en âge, Semigroup, Taux de reproduction de base, Stabilité.

1. Introduction

During the earliest centuries mankind faces ever more challenging environmental and public health problems, such as emergence of new diseases or the emergence of disease into new regions, and the resurgence diseases (tuberculosis, malaria HIV/AIDS, HBV). Mathematical models of populations incorporating age structure, or other structuring of individuals with continuously varing properties, have an extensive history.

The earliest models of age structured populations, due to Sharpe and Lotka in 1911 [37] and McKendrick in 1926 [39] established a foundation for a partial differential equations approach to modeling continuum age structure in an evolving population. At this early stage of development, the stabilization of age structure in models with linear mortality and fertility processes was recognized, although not rigorously established [35, 36]. Rigorous analysis of these linear models was accomplished later in 1941 by Feller [16], in 1963 by Bellman and Cooke [4], and others, using the methods of Volterra integral equations and Laplace transforms. Many applications of this theory have been developed in demography [9, 27, 33, 43], in biology [1, 2, 3, 10, 24, 48] and in epidemiology [7, 8, 17, 18, 22, 32, 13, 12].

The increasingly complex mathematical issues involved in nonlinearities in age structured models led to the development of new technologies, and one of the most useful of these has been the method of semi-groups of linear and nonlinear operators in Banach spaces. Structured population models distinguish individual from another according to characteristics such as age, size, location, status and movement. The goal of structured population is to understand how these characteristics affect the dynamics of these models and thus the outcomes and consequence of the biological and epidemiological processes.

In this paper we consider a mathematical S-I-L (Susceptibles-Infected-Lost of sight) model with demographics process, for the spread of a directly transmitted infectious disease in an age-structured population. By infected (I) we mean infectious taking a chemoprophylaxis in a care center. And by loss of sight (L), we mean infectious that begun their effective therapy in the hospital and never return to the hospital for the spuctrum examinations for many reasons such as long duration of treatment regimen, poverty, mentality, etc... The lost of sight class was previously consider in some papers as [6, 15].

In this paper, the infective agent can be transmitted not only horizontally but also vertically from infected mothers to their newborns (perinatal transmission). There are important infective agents such as HBV (hepatitis B virus), HIV (human immunodeficiency virus) and HTLV (human T-cell leukemia virus) that can be vertically transmitted. Compared with the pure horizontal transmission case, so far we have not yet so many results for vertically diseases in structured populations. In Africa, the vertical transmission of the disease like HIV is in progression nowadays.

Worldwide, 1% of pregnant women are HIV-positive. However, sub-Saharan Africa where 95% of HIV positive women live carries the vast majority of this burden [46]. Without treatment, approximately 25%-50% of HIV-positive mothers will transmit the virus to their newborns during pregnancy, childbirth, or breastfeeding [5]. In 2007, over 2 million children worldwide were living with HIV/AIDS, with the overwhelming majority again in sub-Saharan Africa [46]. Approximately 400,000 infants contract HIV from their mother every year, which is about 15% of the total global HIV incidence [41, 50]. The rate of pediatric HIV infections in sub-Saharan Africa remains unacceptably high, with over 1,000 newborns infected with HIV per day [25].

Large simple trials which aim to study therapeutic interventions and epidemiological associations of human immunodeficiency virus (HIV) infection, including perinatal transmission, in Africa may have substantial rates of lost of sight. A better understanding of the characteristics and the impact of women and children lost of sight is needed. According to Ioannidis et al. [30], for the impact of lost of sight and vertical transmission cohort in Malawi, several predictors of lost of sight were identified in this large HIV perinatal cohort. Lost of sights can impact the observed transmission rate and the risk associations in different studies. They (Ioannidis et al.) also focus that the HIV infection status could not be determine for 36.9% of infant born to HIV-infected mothers; 6.7% with missing status because of various sample problems and 30.3% because they never returned to the clinic (Lost of sight).

Firstly, the epidemic system is formulated. Then, we will describe the semigroup approach to the time evolution problem of the abstract epidemic system. Next we consider the disease invasion process to calculate the basic reproduction ratio R_0, then, we prove that the disease-free steady state is locally asymptotically stable if $R_0 < 1$. Subsequently, we show that at least one endemic steady state exists if the basic reproduction ratio R_0 is greater than unity. By introducing a bifurcation parameter, we show that the endemic steady state is forwardly bifurcated from the disease-free steady state when the basic reproduction ratio crosses unity. Finally, we consider the conditions for the local stability of the endemic steady states.

2. The model

In this section, we formulate a model for the spread of the disease in a host population. We consider a host population divided into three subpopulations; the susceptible class, the infective class (those who are infectious but taking a chemoprophylaxis) and the lost of sight class (those who are infectious but not on a chemoprophylaxis) denoted respectively by $S(t,a)$, $I(t,a)$ and $L(t,a)$ at time t and at specific age a. Let $\beta(.,.)$ be the contact rate between susceptible individuals and infectious individuals. Namely, $\beta(a,\sigma)$ is the transmission rate from the infectious individuals aged σ to the susceptible individuals aged a. All recruitment is into the susceptible class and occur at a specific rate $\Lambda(a)$. The rate of non-disease related death is $\mu(a)$. Infected and lost of sight have additional death rates due to the disease $d_1(a)$ and $d_2(a)$ respectively. The transmission of the disease occurs following adequate contacts between a susceptible and infectious or lost of sight. $r(a)$ denoted the proportion of individuals receiving an effective therapy in a care center and $\phi(a)$ the fraction of them who after begun their treatment will not return in the hospital for the examination. After some time, some of them can return in the hospital at specific rate $\gamma(a)$.

The basic system (age-structured SIL epidemic model) with vertical transmission can be formulated as follows by equation (1).

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)S(t,a) &= \Lambda(a)-(\lambda(t,a)+\mu(a))S(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)I(t,a) &= \lambda(t,a)S(t,a)-(\mu(a)+d_1(a) \\ & \quad +r(a)\phi(a))I(t,a)+\gamma(a)L(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)L(t,a) &= r(a)\phi(a)I(t,a)-(\mu(a)+d_2(a) \\ & \quad +\gamma(a))L(t,a). \end{cases} \tag{1}$$

For the boundary conditions of model (1), we consider that pregnant lost of sight women generally return to the clinic for the birth of they new born, therefore, we can assume that there is not lost of sight new born (i.e. $L(t,0)=0$). Due to the above consideration, the initial boundary conditions of model (1) is given by:

$$\begin{cases} S(t,0) &= \int_0^{a^+} f(a)[S(t,a)+(1-p)(I(t,a)+L(t,a))]da, \\ I(t,0) &= p\int_0^{a^+} f(a)(I(t,a)+L(t,a))da, \\ L(t,0) &= 0, \\ S(0,a) &= \varphi_S(a); \quad a\in(0,a^+), \\ I(0,a) &= \varphi_I(a); \quad a\in(0,a^+), \\ L(0,a) &= \varphi_L(a); \quad a\in(0,a^+), \end{cases} \tag{2}$$

and where $f(a)$ is the age-specific fertility rate, p is the proportion of newborns produced from infected individuals who are vertically infected and $a^+<\infty$ is the upper bound of age. The force of infection $\lambda(t,a)$ is given by

$$\lambda(t,a)=\int_0^{a^+}\beta(a,\sigma)(I(t,\sigma)+L(t,\sigma))d\sigma.$$

where $\beta(a,s)$ is the transmission rate between the susceptible individuals aged a and infectious or lost of sight individuals aged s. $a^+<\infty$ is the upper bound of age.
Let us note that in the literature the transmission rate $\beta(a,\sigma)$ can take many forms: $\beta(a,\sigma)=\beta=constant$ (Dietz 1975 [11]; Greenhalgh 1987 [19]), $\beta(a,\sigma)=g(a)$ (Gripenberg 1983 [20]; Webb 1985 [49]), $\beta(a,\sigma)=g(a)h(\sigma)$ (Dietz and Schenzle 1985 [14]; Greenhalgh 1988 [23]; Castillo-Chavez and al. 1989 [8]).

In the following, we consider systems (1)-(2) under following assumption:

Assumption 1. *We assume that* $\beta\in L^\infty[(0,a^+,\mathbb{R}_+)\times(0,a^+,\mathbb{R}_+)]$ *and functions* $f,\ d_1,\ d_2,\ \gamma,\ \Lambda,\ \mu$ *belong to* $L^\infty(0,a^+,\mathbb{R}_+)$.

3. Existence of flow

The aim of this section is to derive premininary remarks on (1)-(2). These results include the existence of the unique maximal bounded semiflow associated to this system.

3.1. Abstract formulation

Let X be the space defined as

$$X := L^1(0, a^+, \mathbb{R}^3)$$

with the norm

$$||\varphi||_X = \sum_{i=1}^{3} ||\varphi_i||_{L^1};$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in X$. Let us note X_+ the positive cone of X.
It is well known that $(X, ||.||_X)$ is a Banach space. Let $A : D(A) \subset X \to X$ be a operator defined by

$$A\varphi = -\varphi' - \mu\varphi, \tag{3}$$

with the domain

$$\begin{aligned} D(A) \quad = \quad & \left\{ \varphi = (\varphi_1, \varphi_2, \varphi_3) \in W^{1,1}(0, a^+, \mathbb{R}^3) \; and \; \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} = \right. \\ & \left. \begin{pmatrix} \int_0^{a^+} f(a)[\varphi_1(a) + (1-p)(\varphi_2(a) + \varphi_3(a))]da \\ p\int_0^{a^+} f(a)(\varphi_2(a) + \varphi_3(a))da \\ 0 \end{pmatrix} \right\} ; \end{aligned}$$

the function $F : \overline{D(A)} \to X$ defined by

$$F \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \Lambda - \lambda[., \varphi]\varphi_1 \\ \lambda[., \varphi]\varphi_1 - (d_1 + r\phi)\varphi_2 + \gamma\varphi_3 \\ r\phi\varphi_2 - (d_2 + \gamma)\varphi_3 \end{pmatrix},$$

$\lambda[., \varphi] \in L^1(0, a^+, \mathbb{R})$ is a function such that

$$\lambda[a, \varphi] = \int_0^{a^+} \beta(a, \sigma)[\varphi_2(\sigma) + \varphi_3(\sigma)]d\sigma$$

and $W^{1,1}(0, a^+, \mathbb{R})$ is a usual Sobolev space.

Let us first derive the following lemma on operator A.

Lemma 1. *1) The operator A is generator of a C_0-semigroup of linear bounded operators $\{T(t)\}_{t\geq 0}$ such that*

$$T(t)\varphi(a) \quad = \quad \begin{cases} \varphi(a-t) & if \quad a - t \geq 0 \\ C(t-a) & if \quad a - t \leq 0 \end{cases} \qquad for \;\; t \leq a^+$$

and $T(t)\varphi(a) = 0_{\mathbb{R}^3}$ for $t > a^+$; where $C(t) = (C_1(t), C_2(t), 0) \in \mathbb{R}^3$ is the unique solution of the following Volterra integral equation

$$C(t) = G(t) + \Phi(t, C),$$

with

$$G(t) = \left(\int_t^{a^+} f(s)(\varphi_1(s-t) + (1-p)\varphi_2(s-t) + \varphi_3(s-t))ds \; ; \; p\int_t^{a^+} f(s)\varphi_2(s-t)ds \; ; \; 0 \right),$$

$$\Phi(t, C) = \left(\int_0^t f(s)(C_1(t-s) + (1-p)C_2(t-s))ds \; ; \; p\int_0^t f(s)C_2(t-s)ds \; ; \; 0 \right).$$

2) The domain $D(A)$ of operator A is dense in X and A is a closed operator.

Proof. The proof of this result is rather standard. Standard methodologies apply to provide item 1 (see Pazy 1983 [40]). Item 2 is a direct consequence of the fact that the operator A is generator of a C_0-semigroup of linear bounded operators (see Corollary 2.5 in Pazy 1983 [40]). $\square$

Therefore, one obtains that System (1)-(2) re-writes as the following densely defined Cauchy problem

$$\left\{\begin{array}{rcl}\dfrac{d\varphi(t)}{dt} & = & A\varphi(t) + F(\varphi(t)),\\ \varphi(0) & = & (\varphi_S, \varphi_I, \varphi_L)^T.\end{array}\right. \tag{4}$$

3.2. Existence and uniqueness of solutions

We set $X_0 := \overline{D(A)}$ and X_{0+} the positive cone of X_0. In general we can not solve (4) in this strong formulation, if $u_0 \in X_{0+} \setminus D(A)$. So, for arbitrary $\varphi_0 \in X_{0+}$, we solve it in the integrated form:

$$\varphi(t) \quad = \quad \varphi_0 + A\int_0^t \varphi(s)ds + \int_0^t F(\varphi(s))ds \; ; t \geqslant 0. \tag{5}$$

A solution of (5) is called a *mild solution* of the initial value problem (4). So, in the following, we are looking for mild solution of abstract Cauchy-problem (4).

We can easily find that:

Lemma 2. *On Assumption 1, the nonlinear operator F from X to X is continuous and locally Lipschitz.*

Using Lemmas 1 and 2 the main results of this section reads as (see Theorem 1.4 in Pazy 1983[40]).

Theorem 1. *Recalling Assumption 1 and let Lemmas 1 and 2 be satisfied. If $\varphi_0 \in X_{0+} := L^1(0, a^+, \mathbb{R}^3_+)$. Then there exists a unique bounded continuous solution φ to the integrated problem (5) defined on $[0, +\infty)$ with values in X_{0+}.*

4. Equilibria

4.1. Disease-Free Equilibrium (DFE)

The following proposition gives the characteristics of the disease-free equilibrium (DFE) of system (1)-(2).

Let us introduce $l(a) = exp\left(-\int_0^a \mu(s)ds\right)$ the average lifetime of individuals at age a.

Proposition 1. *Let* $\int_0^{a^+} f(a)l(a)da < 1$ *be satisfied. Then, system (1)-(2) has a unique Disease Free Equilibrium (DFE),* $\varphi_0 = (S_0, 0_{L^1}, 0_{L^1})$*, where* S_0 *is given by*

$$\begin{cases} S_0(0) = \dfrac{1}{1 - \int_0^{a^+} f(a)l(a)da} \displaystyle\int_0^{a^+} f(a)l(a) \left(\int_0^a \frac{\Lambda(s)}{l(s)} ds \right) da, \\ S_0(a) = l(a) \left[S_0(0) + \displaystyle\int_0^a \frac{\Lambda(s)}{l(s)} ds \right] \; for \; 0 \leq a \leq a^+. \end{cases} \tag{6}$$

Proof. : φ is an equilibrium of problem (4) if and only if

$$\varphi \in D(A) \text{ and } A\varphi + F(\varphi) = 0_X. \tag{7}$$

For the DFE we have $\varphi_2 = \varphi_3 \equiv 0_{L^1}$. Hence $\lambda[a, \varphi] \equiv 0_{L^1}$. From where the result follows using straightforward computations. □

4.2. Endemic equilibrium (EE)

φ is an endemic equilibrium of (4) if and only if (7) is satisfied. That is,

$$\begin{aligned} \varphi_1(a) &= \varphi_1(0)l(a) \exp\left(-\int_0^a \lambda[\sigma, \varphi] d\sigma \right) \\ &\quad + \int_0^a \frac{l(a)}{l(s)} \exp\left(-\int_s^a \lambda[\sigma, \varphi] d\sigma \right) \Lambda(s) ds; \end{aligned} \tag{8}$$

$$\begin{aligned} \varphi_2(a) &= \int_0^a \frac{l(a)\Gamma_1(a)}{l(s)\Gamma_1(s)} \exp\left(-\int_s^a r(\sigma)\phi(\sigma) d\sigma \right) [\gamma(s)\varphi_3(s) + \lambda[s, \varphi]\varphi_1(s)] \, ds \\ &\quad + \varphi_2(0)l(a)\Gamma_1(a) \exp\left(-\int_0^a r(\sigma)\phi(\sigma) d\sigma \right); \end{aligned} \tag{9}$$

$$\begin{aligned} \varphi_3(a) &= \varphi_3(0)l(a)\Gamma_2(a) \exp\left(-\int_0^a \gamma(\sigma) d\sigma \right) \\ &\quad + \int_0^a \frac{l(a)\Gamma_2(a)}{l(s)\Gamma_2(s)} \exp\left(-\int_s^a \gamma(\sigma) d\sigma \right) r(s)\phi(s)\varphi_2(s) ds; \end{aligned} \tag{10}$$

$$\varphi_1(0) = \int_0^{a^+} f(a)[\varphi_1(a) + (1-p)(\varphi_2(a) + \varphi_3(a))] da; \tag{11}$$

$$\varphi_2(0) = p \int_0^{a^+} f(a)(\varphi_2(a) + \varphi_3(a)) da; \tag{12}$$

$$\varphi_3(0) = 0. \tag{13}$$

where

$$\begin{aligned} \Gamma_1(a) &= \exp\left(-\textstyle\int_0^a (d_1(s) + r(s)\phi(s)) ds \right); \\ \Gamma_2(a) &= \exp\left(-\textstyle\int_0^a (d_2(s) + \gamma(s)) ds \right). \end{aligned}$$

Let us set $\lambda(s) = \lambda[s, \varphi]$ for $s \in [0, a^+)$. Equation (8) re-write as

$$\varphi_1(a) = \varphi_1(0)A_{11}(\lambda, a) + u_1(\lambda, a). \tag{14}$$

Equations (8) and (9) give

$$\varphi_2(a) \quad = \quad \varphi_1(0)A_{21}(\lambda,a)+\varphi_2(0)A_{22}(a)+u_2(\lambda,a). \tag{15}$$

Equations (10), (13) and (14) give

$$\varphi_3(a) \quad = \quad \varphi_1(0)A_{31}(\lambda,a)+\varphi_2(0)A_{32}(\lambda,a)+u_3(\lambda,a); \tag{16}$$

with

$$\begin{aligned}
&A_{11}(\lambda,a)=l(a)\exp\left(-\int_0^a \lambda(\sigma)d\sigma\right);\\
&A_{21}(\lambda,a)=\int_0^a \chi_{21}(a,s)\lambda(s)\exp\left(-\int_0^s \lambda(\sigma)d\sigma\right)ds;\\
&A_{22}(a)=l(a)\Gamma_1(a);\\
&A_{31}(\lambda,a)=\int_0^a \chi_{31}(a,s)\lambda(s)\exp\left(-\int_0^s \lambda(\sigma)d\sigma\right)ds;\\
&A_{32}(a)=l(a)\Gamma_2(a)\int_0^a \frac{\Gamma_1(s)}{\Gamma_2(s)}r(s)\phi(s)ds;\\
&u_1(\lambda,a)=\int_0^a \frac{l(a)}{l(s)}\Lambda(s)\exp\left(-\int_s^a \lambda(\sigma)d\sigma\right)ds;\\
&u_2(\lambda,a)=\int_0^a \frac{l(a)}{l(s)}\Lambda(s)\int_s^a \frac{\Gamma_1(a)}{\Gamma_1(\eta)}\lambda(\eta)\exp\left(-\int_s^\eta \lambda(\sigma)d\sigma\right)ds;\\
&u_3(\lambda,a)=\int_0^a \frac{l(a)\Gamma_2(a)}{l(s)\Gamma_2(s)}r(s)\phi(s)u_2(\lambda,s)ds\\
&+\int_0^a \frac{l(a)\Gamma_1(a)}{l(s)\Gamma_1(s)}\exp\left(-\int_s^a r(\sigma)\phi(\sigma)d\sigma\right)\gamma(s)\varphi_3(s)ds;
\end{aligned}$$

and

$$\chi_{21}(a,s)=l(a)\frac{\Gamma_1(a)}{\Gamma_1(s)};\ \chi_{31}(a,s)=l(a)\int_s^a \frac{\Gamma_2(a)\Gamma_1(\eta)}{\Gamma_2(\eta)\Gamma_1(s)}r(\eta)\phi(\eta)d\eta.$$

From equations (11) and (12), we respectively deduce that

$$\begin{aligned}
&\left(1-\int_0^{a^+} f(a)[A_{11}(\lambda,a)+(1-p)(A_{21}(\lambda,a)+A_{31}(\lambda,a))]da\right)\varphi_1(0)\\
&-(1-p)\varphi_2(0)\int_0^{a^+} f(a)[A_{22}(a)+A_{32}(a)]da=v_1(\lambda);
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&p\varphi_1(0)\int_0^{a^+} f(a)[A_{21}(\lambda,a)+A_{31}(\lambda,a)]da\\
&+\varphi_2(0)\left(p\int_0^{a^+} f(a)[A_{22}(a)+A_{32}(a)]da-1\right)=-v_2(\lambda);
\end{aligned} \tag{18}$$

where

$$\begin{aligned} v_1(\lambda) &= \int_0^{a^+} f(a)[u_1(\lambda,a) + (1-p)(u_2(\lambda,a) + u_3(\lambda,a))]da; \\ v_2(\lambda) &= p\int_0^{a^+} f(a)[u_2(\lambda,a) + u_3(\lambda,a)]da. \end{aligned}$$

Therefore, we find that $\varphi_1(0) = \dfrac{\Delta_1(\lambda)}{\Delta(\lambda)}$ and $\varphi_2(0) = \dfrac{\Delta_2(\lambda)}{\Delta(\lambda)}$; with

$$\Delta(\lambda) = (1-p)p\int_0^{a^+} f(a)[A_{22}(a) + A_{32}(a)]da \times \int_0^{a^+} f(a)[A_{21}(\lambda,a) + A_{31}(\lambda,a)]da$$
$$+\left(1 - \int_0^{a^+} f(a)[A_{11}(\lambda,a) + (1-p)(A_{21}(\lambda,a) + A_{31}(\lambda,a))]da\right) \times$$
$$\left(p\int_0^{a^+} f(a)[A_{22}(a) + A_{32}(a)]da - 1\right);$$
$$\Delta_1(\lambda) = v_1(\lambda)\left(p\int_0^{a^+} f(a)[A_{22}(a) + A_{32}(a)]da - 1\right)$$
$$-(1-p)v_2(\lambda)\int_0^{a^+} f(a)[A_{22}(a) + A_{32}(a)]da;$$
$$\Delta_2(\lambda) = v_2(\lambda)\left(\int_0^{a^+} f(a)[A_{11}(\lambda,a) + (1-p)(A_{21}(\lambda,a) + A_{31}(\lambda,a))]da - 1\right)$$
$$-pv_1(\lambda)\int_0^{a^+} f(a)[A_{21}(\lambda,a) + A_{31}(\lambda,a)]da.$$

Equations (15) and (16) give

$$\left\{\begin{aligned} \varphi_2(a) &= \frac{\Delta_1(\lambda)}{\Delta(\lambda)}A_{21}(\lambda,a) + \frac{\Delta_2(\lambda)}{\Delta(\lambda)}A_{22}(a) + u_2(\lambda,a) \\ \varphi_3(a) &= \frac{\Delta_1(\lambda)}{\Delta(\lambda)}A_{31}(\lambda,a) + \frac{\Delta_2(\lambda)}{\Delta(\lambda)}A_{32}(a) + u_3(\lambda,a) \end{aligned}\right. \tag{19}$$

Since $\lambda(a) = \int_0^{a^+} \beta(a,s)(\varphi_2(s) + \varphi_3(s))ds$; then we have

$$\lambda(a) = H(\lambda)(a); \tag{20}$$

where H is the operator defined from $L^1(0, a^+, \mathbb{R})$ into itself by

$$\begin{aligned} H(\varphi)(a) &= \int_0^{a^+} \beta(a,s)\left[\frac{\Delta_1(\varphi)}{\Delta(\varphi)}(A_{21}(\varphi,s) + A_{31}(\varphi,s)) + u_2(\varphi,s) + u_3(\varphi,s)\right. \\ &\left. + \frac{\Delta_2(\varphi)}{\Delta(\varphi)}(A_{22}(s) + A_{32}(s))\right] ds. \end{aligned} \tag{21}$$

Hence, system (1)-(2) have an endemic equilibrium if and only if the fixed point equation (20) has at least one positive solution.

Now let us introduce the following technical assumptions on the transmission rate β as in Inaba [26, 28, 29]:

Assumption 2. *1) $\beta \in L^1_+(\mathbb{R}\times\mathbb{R})$ such that $\beta(a,s)=0$ for all $(a,s)\notin [o,a^+]\times[0,a^+]$.*

2) $\lim\limits_{h\to 0}\int_{-\infty}^{+\infty}|\beta(a+h,\xi)-\beta(a,\xi)|da=0$ for $\xi\in\mathbb{R}$.

3) It exists a function ε such that $\varepsilon(s)>0$ for $s\in(0,a^+)$ and $\beta(a,s)\geqslant\varepsilon(s)$ for all $(a,s)\in(0,a^+)^2$.

On the above assumption, some properties of operator H are given by the following lemma.

Lemma 3. *Let Assumption 2 be satisfied.*

(i) H is a positive, continu operator. There exist a closed, bounded and convex subset $D\subset L^1_+(0,a^+,\mathbb{R})$ such that $H(D)\subset D$.

(ii) Operator H has a Fréchet derivative H_0 at the point $\varphi\equiv 0$ defined by (22) and $H_0:=H'(0)$ is a positive, compact and nonsupporting operator.

Proof. (i) The positivity and the continuity of operator H are obvious. Let $\varphi\in L^1(0,a^+,\mathbb{R}_+)$, then

$$A_{21}(\varphi,a)\le 1;\;\; A_{31}(\varphi,a)\le\int_0^a\frac{l(a)\Gamma_2(a)}{l(s)\Gamma_2(s)}r(s)\phi(s)ds:=\tilde{A}_{31}(a);$$

$$u_1(\varphi,a)\le\int_0^a\frac{l(a)}{l(s)}\Lambda(s)ds;\;\; u_2(\varphi,a)\le a||\Lambda||_\infty \text{ and}$$

$$u_3(\varphi,a)\le||\Lambda||_\infty\tilde{A}_{31}(a)+\sup_{s\in[0,a]}\gamma(s)||\varphi||_{L^1}.$$

Since $\frac{\Delta_1(\varphi)}{\Delta(\varphi)}=\varphi_1(0)$; $\frac{\Delta_2(\varphi)}{\Delta(\varphi)}=\varphi_2(0)$ and the flow of system (1)-(2) is bounded (Theorem 1), we can find $M_\Omega>0$ such that $|\varphi_1(0)|\le M_\Omega$ and $|\varphi_2(0)|\le M_\Omega$. Therefore, $||H(\varphi)||_{L^1}\le M$; with

$$M=||\beta||_\infty\int_0^{a^+}\left[M_\Omega(1+A_{22}(s)+(\tilde{A}_{31}(s)+A_{32}(s))+\sup_{s\in[0,a]}\gamma(s))+||\Lambda||_\infty(\tilde{A}_{31}(s)+s)\right]ds.$$

Setting $D=\overline{B_+(0,M)}$ with $\overline{B_+(0,M)}:=\{\varphi\in L^1(0,a^+,\mathbb{R}_+):||\varphi||_{L^1}\le M\}$. Hence $H(D)\subset D$. This end the proof of item (i).

(ii) We find that

$$H_0(\psi)(a)=\int_0^{a^+}\beta(a,s)\left[\frac{\Delta_1(0)}{\Delta(0)}(DA_{21}(0,s)(\psi)+DA_{31}(0,s)(\psi))+Du_2(0,s)(\psi)\right.$$
$$\left.+Du_3(0,s)(\psi)+\frac{D\Delta_2(0)(\psi)}{\Delta(0)}(A_{22}(s)+A_{32}(s))\right]ds.$$

where Du denotes the derivative of the function u and

$$Du_2(0,a)(\psi) = \int_0^a \chi_2(a,s)\psi(s)ds; \quad Du_3(0,a)(\psi) = \int_0^a \chi_3(a,s)\psi(s)ds;$$

$$DA_{21}(0,a)(\psi) = \int_0^a \chi_{21}(a,s)\psi(s)ds; \quad DA_{31}(0,a)(\psi) = \int_0^a \chi_{31}(a,s)\psi(s)ds;$$

$$D\Delta_2(0)(\psi) = p\int_0^{a^+} \chi_4(a)\psi(a)da.$$

with

$$\chi_{21}(a,s) = \frac{l(a)\Gamma_1(a)}{l(s)\Gamma_1(s)} \exp\left(-\int_s^a r(\sigma)\phi(\sigma)d\sigma\right) l(s)$$

$$\chi_{31}(a,s) = \int_s^a \frac{l(a)\Gamma_2(a)}{l(\eta)\Gamma_2(\eta)} r(\eta)\phi(\eta)\chi_{21}(\eta,s)d\eta$$

$$\chi_2(a,s) = \chi_{21}(a,s)\int_0^s \frac{\Lambda(\eta)}{l(\eta)}d\eta; \quad \chi_3(a,s) = \chi_{31}(a,s)\int_0^s \frac{\Lambda(\eta)}{l(\eta)}d\eta;$$

$$\chi_4(a) = \left[\frac{S_0(a)}{l(a)}\int_0^{a^+} f(\sigma)l(\sigma)d\sigma - S_0(0)\right]\int_a^{a^+} f(s)\left[\chi_{21}(s,a) + \chi_{31}(s,a)\right]ds.$$

Hence, operator H_0 read as a kernel operator:

$$H_0(\psi)(a) = \int_0^{a^+} \chi(a,s)\psi(s)ds; \tag{22}$$

where the kernel $\chi(a,s)$ is defined by

$$\begin{aligned} \chi(a,s) &= \frac{S_0(s)}{l(s)}\int_s^{a^+} \beta(a,\eta)\left(\chi_{21}(\eta,s) + \chi_{31}(\eta,s)\right)d\eta \\ &\quad + \frac{p\chi_4(s)}{\Delta(0)}\int_0^{a^+} \beta(a,\sigma)(A_{22}(\sigma) + A_{32}(\sigma))d\sigma. \end{aligned} \tag{23}$$

The positivity of H_0 is obvious. Let us show the compactness of the operator H_0 on Assumption 2. Let $\psi \in L^1$ and $\epsilon > 0$. From Assumption 2; there exists $\rho = \rho(\epsilon) > 0$ such that, for $|h| < \rho$ we have $\int_0^{a^+} |\beta(a+h,\xi) - \beta(a,\xi)|da < \epsilon$. Is therefore $h \in \mathbb{R}$ such that $|h| < \rho$. $||\tau_h H_0(\psi) - H_0(\psi)||_{L^1} = \displaystyle\int_0^{a^+} |H_0(\psi)(a+h) - H_0(\psi)(a)|da$. It is easily checked that

$$|H_0(\psi)(a+h) - H_0(\psi)(a)| \leq ||\psi||_{L^1}\int_0^{a^+} |\beta(a+h,s) - \beta(a,s)|C_1(s)ds;$$

where

$$\begin{aligned} C_1(a) &= \left(||\Lambda||_\infty + \frac{\Delta_1(0)}{\Delta(0)}\right)\left(1 + \int_0^a \frac{l(a)\Gamma_2(a)}{l(s)\Gamma_2(s)}r(s)\phi(s)ds\right) \\ &\quad + \frac{||\Lambda||_\infty}{\Delta(0)}(A_{22}(a) + A_{32}(a))\int_0^{a^+} f(a)\left(1 + \int_0^a \frac{l(a)\Gamma_2(a)}{l(s)\Gamma_2(s)}r(s)\phi(s)ds\right)da. \end{aligned}$$

Since $\left(|h| < \rho \Longrightarrow \int_0^{a^+} |\beta(a+h,s) - \beta(a,s)|da < \epsilon\right)$, it comes that

$$||\tau_h H_0(\psi) - H_0(\psi)||_{L^1} \quad \leq \quad \epsilon \left(\int_0^{a^+} C_1(a)da \right) ||\psi||_{L^1}.$$

Let $\mathbb{B}$ a bounded subset of L^1 such that $\psi \in \mathbb{B}$. Then

$$||\tau_h H_0(\psi) - H_0(\psi)||_{L^1} \quad \leq \quad \epsilon \left(\int_0^{a^+} C_1(a)da \right) \times \sup_{\varphi \in \mathbb{B}}\{||\varphi||_{L^1}\}.$$

Applying the Riesz-Fréchet-Kolmogorov theorem on $H_0(\mathbb{B})$ we conclude that $H_0(\mathbb{B})$ is relatively compact. From where H_0 si a compact operator.
Now, let us check that H_0 is a nonsupporting operator. We define the operator $F_0 \in (L^1(0,a^+,\mathbb{R}_+))^*$ (dual space of $L^1(0,a^+,\mathbb{R}_+)$) by

$$\langle F_0; \psi \rangle \quad = \quad \int_0^{a^+} \varepsilon(s)[Du_2(0,s)(\psi) + \delta(s)Du_3(0,s)]ds;$$

where ε is the positive function given by Assumption 2 and $\langle F_0; \psi \rangle$ is the value of $F_0 \in (L^1(0,a^+,\mathbb{R}_+))^*$ at $\psi \in L^1(0,a^+,\mathbb{R}_+)$. Then for $\psi \in L^1(0,a^+,\mathbb{R}_+)$ we have $H_0(\psi) \geq \langle F_0; \psi \rangle \cdot e$ (with $e = 1 \in L^1(0,a^+,\mathbb{R}_+)$). From where $H_0^{n+1}(\psi) \geq \langle F_0; \psi \rangle \langle F_0; e \rangle^n \cdot e \ \ \forall n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}^*$; $F \in (L^1(0,a^+,\mathbb{R}_+))^* \setminus \{0\}$ and $\psi \in L^1(0,a^+,\mathbb{R}_+) \setminus \{0\}$ we have $\langle F; H_0^n(\psi) \rangle > 0$. Therefore, H_0 is a nonsupporting operator. $\square$

The main results of this section reads as

Theorem 2. *Let Assumption 2 be satisfied. Let us note $R_0 = \rho(H_0)$ the spectral radius of operator H_0.*

1) If $R_0 \leq 1$, system (1)-(2) has a unique DFE defined by (6);

2) If $R_0 > 1$, in addition to the DFE, system (1)-(2) has at least one endemic equilibrium.

Proof. The operator H always has $\lambda \equiv 0$ as fixed point. This corresponds to the permanent DFE for system (1)-(2). For the rest, we are looking for the positive fixed point to the operator H. From Lemma 3 we know that there exists a closed, bounded and convex subset D of $L^1(0,a^+,\mathbb{R}_+)$ which is invariant by the operator H. Moreover, from Lemma 3, H has a Fréchet derivative H_0 at the point 0 and $H_0 = DH(0)$ is a compact and nonsupporting operator. Therefore, there exists a unique positive eigenvector ψ_0 corresponding to the eigenvalue $R_0 = \rho(H_0)$ of H_0. Using the same arguments as for the Krasnoselskii fixe point theorem [34], it comes that if $R_0 = \rho(H_0) > 1$, then the operator H has at least one positive fixed point $\lambda^* \in L^1(0,a^+,\mathbb{R}_+) \setminus \{0\}$, corresponding to the EE of system (1)-(2).

Let us suppose that $R_0 = \rho(H_0) \leq 1$. If the operator H has a positive fixe point $\lambda^* \in L^1(0,a^+,\mathbb{R}_+)\setminus\{0\}$ then $\lambda^* = H(\lambda^*)$. Let us notice that $H - H_0 \in L^1(0,a^+,\mathbb{R}_+)\setminus\{0\}$; hence $\lambda^* \leq H_0(\lambda^*)$. Let $F_0 \in (L^1(0,a^+,\mathbb{R}_+))^* \setminus \{0\}$ be the positive eigenfunctional corresponding to the eigenvalue $R_0 = \rho(H_0)$ of H_0 (Sawashima [44]). Therefore

$$\begin{aligned} 0 \leq \langle F_0; H_0(\lambda^*) - \lambda^* \rangle \quad &= \quad \langle F_0, ; H_0(\lambda^*) \rangle - \langle F_0; \lambda^* \rangle\,; \\ &= \quad \rho(H_0)\,\langle F_0; \lambda^* \rangle - \langle F_0; \lambda^* \rangle\,; \\ &= \quad (\rho(H_0) - 1)\,\langle F_0; \lambda^* \rangle\,. \end{aligned}$$

From where $(\rho(H_0)-1)\langle F_0;\lambda^*\rangle \geq 0$. Since $\langle F_0;\lambda^*\rangle > 0$, it follows that $\rho(H_0)\geq 1$; which is a contradiction. $\square$

5. Stability analysis for equilibrium

In order to investigate the local stability of the equilibrium solutions $(S^*(a); I^*(a); L^*(a))$ we rewrite (1)-(2) into the equation for small perturbations. Let

$$(S(t,a), I(t,a), L(t,a)) = (S^*(a), I^*(a), L^*(a)) + (x(t,a), y(t,a), z(t,a)).$$

Then from system (1) we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)x(t,a) &= -\lambda(t,a)(S^*(a)+x(t,a)) \\
&\quad -(\mu(a)+\lambda^*(a))x(t,a); \qquad (24)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)y(t,a) &= \lambda(t,a)(x(t,a)+S^*(a))+\lambda^*(a)x(t,a) \\
&\quad -(\mu(a)+d_1(a)+r(a)\phi(a))y(t,a); \qquad (25)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)z(t,a) &= r(a)\phi(a)y(t,a)-(\mu(a)+d_2(a))z(t,a); \qquad (26)
\end{aligned}$$

and from (2) we also have

$$\left\{\begin{array}{lcl}
x(t,0) &=& \int_0^{a^+} f(a)[x(t,a)+(1-p)(y(t,a)+z(t,a))]da;\\
y(t,0) &=& p\int_0^{a^+} f(a)(y(t,a)+z(t,a))da;\\
z(t,0) &=& 0;
\end{array}\right. \qquad (27)$$

with $\lambda(a,t) = \displaystyle\int_0^{a^+}\beta(a,s)(y(t,s)+z(t,s))ds$ and $\lambda^*(a) = \displaystyle\int_0^{a^+}\beta(a,s)(I^*(s)+L^*(s))ds$.

Let us note $u(t)=(x(t),y(t),z(t))^T$. Then from equations (24), (25) and (26) we have

$$\frac{d}{dt}u(t) = Au(t)+G(u(t)); \qquad (28)$$

where A is the operator defined by (3). The nonlinear term G is defined by

$$G(u) = \begin{pmatrix} -\mathcal{P}(u_2,u_3)(u_1+S^*)-(\lambda^*+\mu)u_1 \\ \mathcal{P}(u_2,u_3)(u_1+S^*)+\lambda^*u_1-(\mu+d_1+r\phi)u_2 \\ r\phi u_2-(\mu+d_2)u_3 \end{pmatrix};$$

and $\mathcal{P}$ is linear operator defined on $L^1\times L^1$ by

$$\mathcal{P}(u_2,u_3)(a) = \int_0^{a^+}\beta(a,s)(u_2(s)+u_3(s))ds. \qquad (29)$$

The linearized equation of (28) around $u = 0$ is given by

$$\frac{d}{dt}u(t) = (A + C)u(t); \tag{30}$$

where the linear operator C is the Fréchet derivative of $G(u)$ at $u = 0$ and it is given by

$$C(u) = \begin{pmatrix} -\mathcal{P}(u_2, u_3)S^* - (\lambda^* + \mu)u_1 \\ \mathcal{P}(u_2, u_3)S^* + \lambda^* u_1 - (\mu + d_1 + r\phi)u_2 \\ r\phi u_2 - (\mu + d_2)u_3 \end{pmatrix}$$

Now let us consider the resolvent equation for $\widehat{A} + C$:

$$(z - (A + C))\psi = \vartheta; \quad \psi \in D(A), \quad \vartheta \in X, \quad z \in \mathbb{C}. \tag{31}$$

Applying the variation of constant formula to (31) we obtain the following equations:

$$\psi_1(a) = \Pi(a)l(a)e^{-za}\left[\psi_1(0) + \int_0^a (T_{11}(s)\vartheta_1(s) - T_{12}(s)\mathcal{P}(\psi_1, \psi_2)(s))ds\right]; \tag{32}$$

$$\begin{aligned}\psi_2(a) &= \left[\psi_2(0) + \int_0^a \frac{e^{zs}}{\Gamma_1(s)l(s)}(\vartheta_2(s) + \lambda^*(s)\psi_1(s) + \mathcal{P}(\psi_1, \psi_2)(s)S^*(s))ds\right] \\ &\quad \times \Gamma_1(a)l(a)e^{-za};\end{aligned} \tag{33}$$

$$\psi_3(a) = \Gamma_2(a)l(a)e^{-za}\left[\psi_3(0) + \int_0^a \frac{e^{zs}}{\Gamma_2(s)l(s)}(\vartheta_3(s) + r(s)\phi(s)\psi_2(s))ds\right]. \tag{34}$$

with $\Pi(a) = \exp\left(-\int_0^a \lambda^*(\sigma)d\sigma\right)$; $T_{11}(s) = \dfrac{e^{zs}}{\Pi(s)l(s)}$ and $T_{12}(s) = S^*(s)T_{11}(s)$.
Equations (32)-(33) and (35)-(34) respectively gives

$$\begin{aligned}\psi_2(a) &= \Gamma_1(a)l(a)e^{-za}\left[\psi_2(0) + T_{21}(a)\psi_1(0) + \int_0^a T_{23}(z, a, s)\mathcal{P}(\psi_1, \psi_2)(s))ds\right. \\ &\quad \left. + \int_0^a T_{24}(z, a, s)\vartheta_1(s)ds + \int_0^a T_{25}(z, s)\vartheta_2(s)ds\right]\end{aligned} \tag{35}$$

and

$$\begin{aligned}\psi_3(a) = \Gamma_2(a)l(a)e^{-za}&\left[T_{32}(a)\psi_2(0) + T_{31}(a)\psi_1(0) + \psi_3(0) + \int_0^a T_{33}(z, a, s)\mathcal{P}(\psi_1, \psi_2)(s))ds\right. \\ &\left. + \int_0^a T_{34}(z, a, s)\vartheta_1(s)ds + \int_0^a T_{35}(z, a, s)\vartheta_2(s)ds + \int_0^a T_{36}(z, a, s)\vartheta_3(s)ds\right];\end{aligned} \tag{36}$$

where

$$T_{21}(a) = \int_0^a \frac{\Pi(s)}{\Gamma_1(s)}\lambda^*(s)ds; \quad T_{24}(z, a, s) = \frac{e^{zs}}{l(s)\Pi(s)}\int_s^a \frac{\Pi(\sigma)}{\Gamma_1(\sigma)}\lambda^*(\sigma)d\sigma,$$

$$T_{23}(z, a, s) = \frac{e^{zs}}{l(s)}S^*(s)\left(\frac{1}{\Gamma_1(s)} - \frac{1}{\Pi(s)}\int_s^a \frac{\Pi(\sigma)}{\Gamma_1(\sigma)}\lambda^*(\sigma)d\sigma\right),$$

$$T_{25}(z,s)=\frac{e^{zs}}{l(s)\Gamma_1(s)},\ \ T_{31}(a)=\int_0^a\frac{\Gamma_1(s)}{\Gamma_2(s)}r(s)\phi(s)T_{21}(s)ds,$$
$$T_{32}(a)=\int_0^a\frac{\Gamma_1(s)}{\Gamma_2(s)}r(s)\phi(s)ds,\ \ T_{36}(z,a)=\frac{e^{za}}{\Gamma_2(a)l(a)},$$
$$T_{33}(z,a,s)=\int_s^a\frac{\Gamma_1(\sigma)}{\Gamma_2(\sigma)}r(\sigma)\phi(\sigma)T_{23}(z,\sigma,s)d\sigma,$$
$$T_{34}(z,a,s)=\int_s^a\frac{\Gamma_1(\sigma)}{\Gamma_2(\sigma)}r(\sigma)\phi(\sigma)T_{24}(z,\sigma,s)d\sigma,$$
$$T_{35}(z,a,s)=T_{25}(z,s)\int_s^a\frac{\Gamma_1(\sigma)}{\Gamma_2(\sigma)}r(\sigma)\phi(\sigma)d\sigma.$$

Since $\psi\in D(A)$; it comes that

$$\psi_1(0)=\int_0^{a^+}f(a)[\psi_1(a)+(1-p)(\psi_2(a)+\psi_3(a))]da;\tag{37}$$
$$\psi_2(0)=p\int_0^{a^+}f(a)(\psi_2(a)+\psi_3(a))da;\tag{38}$$
$$\psi_3(0)=0.\tag{39}$$

Equations (36)-(39); (32)-(35)-(40)-(37) and (35)-(40)-(38) respectively lead to

$$\psi_3(a)=\Gamma_2(a)l(a)e^{-za}\Big[T_{32}(a)\psi_2(0)+T_{31}(a)\psi_1(0)++\int_0^aT_{33}(z,a,s)\mathcal{P}(\psi_1,\psi_2)(s))ds$$
$$+\int_0^aT_{34}(z,a,s)\vartheta_1(s)ds+\int_0^aT_{35}(z,a,s)\vartheta_2(s)ds+\int_0^aT_{36}(z,s)\vartheta_3(s)ds\Big];\tag{40}$$

$$(B_{11}(z)-1)\psi_1(0)+(1-p)B_{12}(z)\psi_2(0)+\int_0^{a^+}B_{13}(z,a)\mathcal{P}(\psi_1,\psi_2)(a)da$$
$$+\int_0^{a^+}B_{14}(z,a)\vartheta_1(a)da+\int_0^{a^+}B_{15}(z,a)\vartheta_2(a)da+\int_0^{a^+}B_{16}(z,a)\vartheta_3(a)da=0;\tag{41}$$

and

$$pB_{21}(z)\psi_1(0)+(pB_{22}(z)-1)\psi_2(0)+p\int_0^{a^+}B_{23}(z,a)\mathcal{P}(\psi_1,\psi_2)(a)da$$
$$+p\int_0^{a^+}B_{24}(z,a)\vartheta_1(a)da+p\int_0^{a^+}B_{25}(z,a)\vartheta_2(a)da+p\int_0^{a^+}B_{26}(z,a)\vartheta_3(a)da=0;\tag{42}$$

with

$$B_{11}(z)=\int_0^{a^+}f(a)l(a)e^{-za}\left[\Pi(a)+(1-p)(\Gamma_1(a)T_{21}(a)+\Gamma_2(a)T_{31}(a)\right]da;$$

$$B_{12}(z) = \int_0^{a^+} f(a)l(a)e^{-za}\left[\Gamma_1(a) + \Gamma_2(a)T_{32}(a)\right] da;$$

$$B_{13}(z,a) = \int_a^{a^+} f(s)l(s)e^{-zs}\left[-T_{12}(a)\Pi(s) + (1-p)(\Gamma_1(s)T_{23}(z,s,a) + \Gamma_2(s)T_{33}(z,s,a))\right] ds;$$

$$B_{14}(z,a) = \int_a^{a^+} f(s)l(s)e^{-zs}\left[T_{11}(a)\Pi(s) + (1-p)(\Gamma_1(s)T_{24}(z,s,a) + \Gamma_2(s)T_{34}(z,s,a))\right] ds;$$

$$B_{15}(z,a) = \int_a^{a^+} f(s)l(s)e^{-zs}\left[\Gamma_1(s)T_{25}(z,a) + (1-p)\Gamma_2(s)T_{35}(z,s,a)\right] ds;$$

$$B_{16}(z,a) = (1-p)\int_a^{a^+} f(s)l(s)e^{-zs}\Gamma_2(s)T_{36}(z,s)ds;$$

$$B_{21}(z) = \int_0^{a^+} f(a)l(a)e^{-za}[\Gamma_1(a)T_{21}(a) + \Gamma_2(a)T_{31}(a)]da;$$

$$B_{22}(z) = \int_0^{a^+} f(a)l(a)e^{-za}[\Gamma_1(a) + \Gamma_2(a)T_{32}(a)]da;$$

$$B_{23}(z,a) = \int_a^{a^+} f(s)l(s)e^{-zs}[\Gamma_1(s)T_{23}(z,s,a) + \Gamma_2(s)T_{33}(z,s,a)]ds;$$

$$B_{24}(z,a) = \int_a^{a^+} f(s)l(s)e^{-zs}[\Gamma_1(s)T_{24}(z,s,a) + \Gamma_2(s)T_{34}(z,s,a)]ds;$$

$$B_{25}(z,a) = T_{25}(z,a)\int_a^{a^+} f(s)l(s)e^{-zs}[\Gamma_1(s)T_{25}(z,a) + \Gamma_2(s)T_{35}(z,s,a)]ds;$$

$$B_{26}(z,a) = T_{36}(z,a)\int_a^{a^+} f(s)l(s)\Gamma_2(s)e^{-zs}ds.$$

System (41)-(42) is a linear system with respect to $\psi_1(0)$ and $\psi_2(0)$, hence

$$\begin{aligned}\psi_1(0) &= \int_0^{a^+} det_{11}(z,a)\mathcal{P}(\psi_1,\psi_2)(a)da + \int_0^{a^+} det_{12}(z,a)\vartheta_1(a)da + \\ &+ \int_0^{a^+} det_{13}(z,a)\vartheta_2(a)da + \int_0^{a^+} det_{14}(z,a)\vartheta_3(a)da; \qquad (43)\\ \psi_2(0) &= \int_0^{a^+} det_{21}(z,a)\mathcal{P}(\psi_1,\psi_2)(a)da + \int_0^{a^+} det_{22}(z,a)\vartheta_1(a)da \\ &+ \int_0^{a^+} det_{23}(z,a)\vartheta_2(a)da + \int_0^{a^+} det_{24}(z,a)\vartheta_3(a)da; \qquad (44)\end{aligned}$$

where

$$det_{11}(z,a) = \frac{-1}{det}\left[(pB_{22}(z)-1)B_{13}(z,a) - p(1-p)B_{12}(z)B_{23}(z,a)\right];$$

$$det_{12}(z,a) = \frac{-1}{det}\left[(pB_{22}(z)-1)B_{14}(z,a) - p(1-p)B_{12}(z)B_{24}(z,a)\right];$$

$$det_{13}(z,a) = \frac{-1}{det}\left[(pB_{22}(z)-1)B_{15}(z,a) - p(1-p)B_{12}(z)B_{25}(z,a)\right];$$

$$det_{14}(z,a) = \frac{-1}{det}\left[(pB_{22}(z)-1)B_{16}(z,a) - p(1-p)B_{12}(z)B_{26}(z,a)\right];$$

$$det_{21}(z,a) = \frac{p}{det}\left[(B_{21}(z)B_{13}(z,a) - (B_{11}(z)-1)B_{23}(z,a)\right];$$

$$det_{22}(z,a) = \frac{p}{det}\left[(B_{21}(z)B_{14}(z,a) - (B_{11}(z)-1)B_{24}(z,a)\right];$$

$$det_{23}(z,a) = \frac{p}{det}\left[(B_{21}(z)B_{15}(z,a) - (B_{11}(z)-1)B_{25}(z,a)\right];$$

$$det_{24}(z,a) = \frac{p}{det}\left[(B_{21}(z)B_{16}(z,a) - (B_{11}(z)-1)B_{26}(z,a)\right];$$

$$det = (B_{11}(z)-1)(pB_{22}(z)-1) - p(1-p)B_{21}(z)B_{12}(z).$$

From equations (29)-(35)-(40)-(43)-(44) it follows that

$$\mathcal{P}(\psi_2,\psi_3)(\eta) = (I-V_z)^{-1}\left[(U_z\vartheta_1)(\eta) + (W_z\vartheta_2)(\eta) + (Y_z\vartheta_3)(\eta)\right]; \tag{45}$$

where V_z, U_z, W_z and Y_z are the Volterra operator define on $L^1(0,a^+,\mathbb{R})$ into itself by

$$\begin{aligned}(U_z\varphi)(a) &= \int_0^{a^+}\Theta_z(\eta,a)\varphi(a)da;\ \ (V_z\varphi)(a) = \int_0^{a^+}\chi_z(\eta,a)\varphi(a)da;\\ (Y_z\varphi)(a) &= \int_0^{a^+}E_z(\eta,a)\varphi(a)da;\ \ (W_z\varphi)(a) = \int_0^{a^+}K_z(\eta,a)\varphi(a)da;\end{aligned} \tag{46}$$

where

$$\begin{aligned}&\chi_z(\eta,a) = C_1^{te}(\eta)det_{11}(z,a) + C_2^{te}(\eta)det_{21}(z,a)\\ &+\int_a^{a^+}\beta(\eta,s)l(s)e^{-zs}[\Gamma_1(s)T_{23}(z,s,a) + \Gamma_2(s)T_{33}(z,s,a)]ds;\end{aligned} \tag{47}$$

$$\begin{aligned}&\Theta_z(\eta,a) = C_1^{te}(\eta)det_{12}(z,a) + C_2^{te}(\eta)det_{22}(z,a)\\ &+\int_a^{a^+}\beta(\eta,s)l(s)e^{-zs}[\Gamma_1(s)T_{24}(z,s,a) + \Gamma_2(s)T_{34}(z,s,a)]ds;\end{aligned}$$

$$\begin{aligned}&K_z(\eta,a) = C_1^{te}(\eta)det_{13}(z,a) + C_2^{te}(\eta)det_{23}(z,a)\\ &+\int_a^{a^+}\beta(\eta,s)l(s)e^{-zs}[\Gamma_1(s)T_{25}(z,s,a) + \Gamma_2(s)T_{35}(z,s,a)]ds;\end{aligned}$$

$$E_z(\eta,a) = C_1^{te}(\eta)det_{14}(z,a) + C_2^{te}(\eta)det_{24}(z,a) + \int_a^{a^+}\beta(\eta,s)l(s)e^{-zs}\Gamma_2(s)T_{36}(z,s,a)ds;$$

and

$$C_1^{te}(\eta) = \int_0^{a^+} \beta(\eta,a)l(a)e^{-za}[\Gamma_1(a)T_{21}(a)+\Gamma_2(a)T_{31}(a)]da;$$
$$C_2^{te}(\eta) = \int_0^{a^+} \beta(\eta,a)l(a)e^{-za}[\Gamma_1(a)+\Gamma_2(a)T_{32}(a)]da;$$

Let us recall some definitions related to a C_0-semi-group $\{T(t)\}_{t\geqslant 0}$ on a Banach space with infinitesimal generator R. The *type* or the *growth bound* of the semi-group $\{T(t)\}_{t\geqslant 0}$ is the quantity:

$$\begin{aligned}&\omega_0(R) := \\ &\inf\{\alpha\in\mathbb{R}:\ \exists M\geq 1 \text{ such that } ||T(t)||\leq Me^{\alpha t}\ \forall t\geq 0\} \\ &= \lim_{t\to 0}\frac{\ln ||T(t)||}{t}.\end{aligned}$$

The *spectral bound* of the C_0-semi-group $\{T(t)\}_{t\geqslant 0}$ is the quantity:

$$s(R) := \sup\{R_e\lambda:\ \lambda\in\sigma_p(R)\},$$

where $\sigma_p(R)$ denote the point spectrum of R.

Wow, we conclude that

Lemma 4. *Recalling Assumptions 1 and 2. Then*

1) The perturbated operator $A+C$ has a compact resolvent and

$$\sigma(A+C)=\sigma_p(A+C)=\{z\in\mathbb{C}:\ 1\in\sigma_p(V_z)\};$$

where $\sigma(A)$ and $\sigma_p(A)$ denote the spectrum of A and the point spectrum of A respectively.

2) Let $\{U(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $A+C$. Then $\{U(t)\}$, $t\geq 0$ is eventually compact and

$$\omega_0(A+C)=s(A+C).$$

Proof. 1) From equations (32), (43) and (46) we find that

$$\psi_1(a) = \Pi(a)l(a)e^{-za}\psi_1(0)+J_1(\vartheta_1)(a)+K_1(\vartheta_1,\vartheta_2)(a);$$

with

$$J_1(\vartheta_1)(a) = \int_0^a \Pi(a)l(a)T_{11}(s)e^{-zs}\vartheta_1(s)ds;$$
$$\begin{aligned}K_1(\vartheta_1,\vartheta_2)(a) = &\int_0^a \Pi(a)l(a)T_{11}(s)S^*(s)e^{-zs}(I-V_z)^{-1} \\ &[(U_z\vartheta_1)(s)+(W_z\vartheta_2)(s)+(Y_z\vartheta_3)(s)]ds.\end{aligned}$$

ψ_1 is a compact operator if and only if J_1 and K_1 are compact. Since J_1 is a Volterra operator with continue kernel, we deduce that J_1 is a compact operator on L^1. Using the same arguments as for the proof of the compactness of operator H_0 (Lemma 3), we can show that the operators U_z, W_z and Y_z are compact for all $z \in \mathbb{C}$. Let us set $\Sigma := \{z \in \mathbb{C} : \ 1 \in \sigma_p(V_z)\}$. Hence, if $z \in \mathbb{C}\setminus\Sigma$ then, K_1 is a compact operator from $L^1 \times L^1$ to L^1. In the same way, we can show that $\psi_2(a)$ and $\psi_3(a)$ are represent by a compact operators. Therefore, the resolvent of $A + C$ is compact. From where $\sigma(A + C) = \sigma_p(A + C)$ (see Kato, p.187 [31]) i.e. $\mathbb{C} \setminus \Sigma \subset \rho(A+C)$ and $\rho(A+C)$ denotes the resolvent of $A+C$. In other words $\Sigma \supset \sigma(A + C) = \sigma_p(A + C)$. Since V_z is a compact operator, we know that $\sigma(V_z) \setminus \{0\} = \sigma_p(V_z) \setminus \{0\}$. If $z \in \Sigma$, then it exists $\psi_z \in L^1 \setminus \{0\}$ such that $V_z\psi_z = \psi_z$. Let us set

$$\phi_1(a) = \Pi(a)l(a)e^{-za}\left[\int_0^{a^+} det_{11}(z,a)\psi_z(a)da - \int_0^a \frac{e^{za}}{\Pi(s)l(s)}\psi_z(s)ds\right];$$

$$\phi_2(a) = \Pi(a)l(a)e^{-za}\left[\int_0^{a^+} det_{21}(z,a)\psi_z(a)da - \int_0^a \frac{e^{za}}{\Gamma_1(s)l(s)}(\lambda^*(s)\phi_1(s) + S^*(s)\psi_z(s))ds\right];$$

$$\phi_3(a) = \Gamma_2(a)l(a)e^{-za}\int_0^a \frac{e^{za}}{\Gamma_2(s)l(s)}r(s)\phi(s)\psi_2(s)ds.$$

Then $(\phi_1, \phi_2, \phi_3)^T$ is an eigenvector of $A + C$ associated to the eigenvalue z. Hence, $z \in \sigma(A + C) = \sigma_p(A + C)$ i.e. $\Sigma \subset \sigma(A + C) = \sigma_p(A + C)$. This end the proof of item 1.

2) For $\psi \in X$, let us set

$$\begin{aligned} C_1\psi &= (-P(\psi_2, \psi_3)S^*, \mathcal{P}(\psi_2, \psi_3)S^*, 0)^T; \\ C_2\psi &= (-(\lambda^* + \mu)\psi_1, \lambda^*\psi_1 - (\mu + d_1 + r\phi)\psi_2 r\phi\psi_2 - (\mu + d_2)\psi_3)^T;. \end{aligned}$$

Then $C = C_1+C_2$. The operator $A+C_2$ generated a nilpotent C_0-semigroup $\{S_2(t)\}_{t\geq 0}$, from where $\{S_2(t)\}_{t\geq 0}$ is norm continuous. Using Assumptions 1 and 2, we find that C_1 is compact operator on X. From Theorem 1.30 of Nagel(1986) [42] it comes that C_1 is generator of a norm continuous C_0-semigroup $\{S_1(t)\}_{t\geq 0}$. Therefore, $S_1(t) + S_2(t)$ is a C_0-semigroup generated by $A + C$ and it is norm continuous (Spectral theorem P.87 Nagel [42]). □

Let us remark that if $\omega_0(A + C) < 0$, the equilibrium $u = 0$ of system (28) is locally asymptotically stable (linearized stability, Webb 1985[49]). Therefore, to study the stability of equilibrium states, we have to know the structure of the set $\Sigma := \{z \in \mathbb{C} : 1 \in \sigma_p(V_z)\}$. Since $||V_z||_{L^1} \to 0$ if $z \to +\infty$, $I - V_z$ is inversible for the large values of $R_e z$.
By theorem of Steinberg(1968)[47], the function $z \rightarrowtail (I - V_z)^{-1}$ is meromorphic in the complex domain, and hence the set Σ is a discrete set whose elements are poles of $(I - V_z)^{-1}$ of finite order.

In the following, we will use elements of positive operator theory.

For the positivity of operator V_z we make the following assumption

Assumption 3.

$$\int_0^{a^+}(d_1(\sigma) + r(\sigma)\phi(\sigma))d\sigma \quad \leq \quad \exp\left(-\int_0^{a^+}\lambda^*(\sigma)d\sigma\right); \tag{48}$$

where $\lambda^*(\sigma) = \int_0^{a^+}\beta(\sigma,\eta)(I^*(\eta) + L^*(\eta))d\eta$.

Lemma 5. *Let Assumption 3 be satisfied. Then*

1) The operator V_z, $z \in \mathbb{R}$, is nonsupporting with respect to $L^1(0, a^+, \mathbb{R}_+)$ and

$$\lim_{z \to -\infty} \rho(V_z) = +\infty \quad ; \quad \lim_{z \to +\infty} \rho(V_z) = 0.$$

2) There exists a unique $z_0 \in \mathbb{R} \cap \Sigma$ such that

$$\rho(V_{z_0}) = 1 \quad and \quad \begin{cases} z_0 > 0 & if \quad \rho(V_0) > 1, \\ z_0 = 0 & if \quad \rho(V_0) = 1, \\ z_0 < 0 & if \quad \rho(V_0) < 1. \end{cases}$$

3) $z_0 > \sup\{R_e z : \ z \in \Sigma \setminus \{z_0\}\}$.

Proof. 1) Let $z \in \mathbb{R}$. Unconditionally, V_z is a positive operator when $\lambda^*(a) \equiv 0$ (case of DFE). When $\lambda^*(a) > 0$, V_z is a positive operator once $\Gamma_1(s)T_{23}(z, a, s) + \Gamma_2(s)T_{33}(z, a, s) \geq 0$ for all $0 \leq a \leq s \leq a^+$. To have the previous inequality, it suffices that inequality (48) of Assumption 3 holds. We can checked that

$$V_z\psi \geq \langle f_z, \psi \rangle \cdot e; \tag{49}$$

where $\psi \in L^1(0, a^+, \mathbb{R}_+)$; $e \equiv 1 \in L^1(0, a^+, \mathbb{R}_+)$ and f_z is a positive linear functional defined by

$$< f_z, \psi > \quad = \quad m \int_0^{a^+} \int_a^{a^+} e^{-z(a-s)} \frac{l(s)}{l(a)} \left(\frac{1}{\Gamma_1(a)} - \frac{1}{\Pi(a)} \int_a^s \frac{\Pi(\sigma)}{\Gamma_1(\sigma)} \lambda^*(\sigma) d\sigma \right) dsda;$$

with $m = \inf_{(a,s) \in [0,a^+)^2} \beta(a, s)$. From (49), we show that $V_z^{n+1}\psi \geq \langle f_z, \psi \rangle \langle f_z, e \rangle^n \cdot e$ for all $n \in \mathbb{N}$. Since f_z is positive operator and $e \in L^1(0, a^+, \mathbb{R}_+) \setminus \{0\}$, we have $\langle F, V_z^n\psi \rangle > 0 \ \forall \psi \in (L^1(0, a^+, \mathbb{R}_+))^* \setminus \{0\} \ \forall \psi \in L^1(0, a^+, \mathbb{R}_+) \setminus \{0\}$. That is V_z is nonsupporting.

Let F_z be the eigenfunctional of V_z that corresponds to the eigenvalue $\rho(V_z)$. Taking the duality pairing into inequality (49), we have

$$\rho(V_z) \langle F_z, \psi \rangle \quad \geq \quad \langle f_z, \psi \rangle \langle F_z, e \rangle .$$

Taking $\psi = e$ and since F_z is positive, it follows that $\rho(V_z) \geq \langle f_z, e \rangle \to +\infty$ when $z \to -\infty$. From where $\lim_{z \to -\infty} \rho(V_z) = +\infty$. since $||V_z||_{L^1} \to 0$ when $z \to +\infty$, we deduce that $\lim_{z \to +\infty} \rho(V_z) = 0$. This end the proof of item 1.

2) Let $h : \mathbb{R} \to \mathbb{C}$; $z \mapsto \rho(V_z)$. The kernel χ_z defined by (47) is strictly decreasing with respect to $z \in \mathbb{R}$. Let $z_1, z_2 \in \mathbb{R}$ such that $z_1 < z_2$, then $\chi_{z_1} < \chi_{z_2}$ that is $V_{z_1} > V_{z_2}$. Since V_{z_1} and V_{z_2} are compact and nonsupporting operators we deduce from Marek(1970) [38] that $\rho(V_{z_1}) > \rho(V_{z_2})$. Therefore, the function h is strictly decreasing. The limits of the function $h(z) = \rho(V_z)$ at $-\infty$ and $+\infty$ give that there exist a unique $z_0 \in \mathbb{R} \cap \Sigma$ such that $\rho(V_{z_0}) = 1$. If $\rho(V_0) > 1$ then $h(0) > h(z_0)$ i.e. $z_0 < 0$ (strictly decreasing of h) and the other cases is show in the same way. This end the proof of item 2.

3)Let $z \in \Sigma$, then there exists $\psi_z \in L^1$ such that $V_z\psi_z = \psi_z$. Let $|\psi_z|$ be a function defined by $|\psi_z|(s) := |\psi_z(s)|$. The definition of V_z leads to

$$|\psi_z| = |V_z\psi_z| \leq V_{R_e z}|\psi_z|. \tag{50}$$

Let $F_{R_e z}$ be the positive eigenfunction associated to the eigenvalue $\rho(V_{R_e z})$ of $V_{R_e z}$. From (50) we deduce that $\langle F_{R_e z}, |\psi_z|\rangle \leq \langle F_{R_e z}, V_{R_e z}|\psi_z|\rangle = r(V_{R_e z})\langle F_{R_e z}, |\psi_z|\rangle$. The positivity of $F_{R_e z}$ implies that $r(V_{R_e z}) \geq 1$ that is $h(R_e z) \geq h(z_0)$ i.e. $z_0 \leq R_e z$. To end the proof, let us show that: if $z_0 = R_e z$ then $z = z_0$.
We know that $|\psi_z| \leq V_{R_e z}|\psi_z| = V_{z_0}|\psi_z|$. Let us suppose that $|\psi_z| < V_{z_0}|\psi_z|$; taking the pairing product with the dual function F_0 corresponding to the eigenvalue $\rho(V_{z_0}) = 1$, one has $\langle F_0, |\psi_z|\rangle > \langle F_0, |\psi_z|\rangle$, which is a contradiction. Hence $|\psi_z| = V_{z_0}|\psi_z|$. Therefore $|\psi_z| = c\psi_0$ where c is constant not equal to zero (Sawashima 1964 [44]) and ψ_0 is the eigenfunction corresponding to $\rho(V_{z_0}) = 1$. So $\psi_z(a) = c\psi_0(a)e^{i\alpha(a)}$ for a reel function α; moreover $|V_z\psi_z| = |\psi_z| = c\psi_0 = cV_{z_0}\psi_0$. Substituting $\psi_z(a) = c\psi_0(a)e^{i\alpha(a)}$ into the equality $|V_z\psi_z| = cV_{z_0}\psi_0$ one has

$$\int_0^{a^+}\int_a^{a^+} \beta(\eta,s)l(s)e^{-z_0(s-a)}[\Gamma_1(s)\tilde{T}_{23}(s,a) + \Gamma_2(s)\tilde{T}_{33}(s,a)]\psi_0(a)dsda =$$
$$\left|\int_0^{a^+}\int_a^{a^+} \beta(\eta,s)l(s)e^{-(z_0+i(s-a)Imz)}[\Gamma_1(s)\tilde{T}_{23}(s,a) + \Gamma_2(s)\tilde{T}_{33}(s,a)]e^{i\alpha(a)}\psi_0(a)dsda\right|; \tag{51}$$

with

$$\begin{aligned}
\tilde{T}_{23}(a,s) &= \frac{S^*(s)}{l(s)}\left(\frac{1}{\Gamma_1(s)} - \frac{1}{\Pi(s)}\int_s^a \frac{\Pi(\sigma)}{\Gamma_1(\sigma)}\lambda^*(\sigma)d\sigma\right); \\
\tilde{T}_{33}(a,s) &= \int_s^a \frac{\Gamma_1(\sigma)}{\Gamma_2(\sigma)}r(\sigma)\phi(\sigma)\tilde{T}_{23}(a,\sigma)d\sigma.
\end{aligned}$$

Applying two times, Lemma 6.12 of Heijmans(1986) [21], to the relation (51) it comes that $(s-a)Imz + \alpha(a) = b$ for all $0 \leq a \leq s \leq a^+$ where b is a constant. From the equality $V_z\psi_z = \psi_z$ one has $e^{ib}V_{z_0}\psi_0 = \psi_0 e^{i\alpha(a)}$ i.e. $b = \alpha(a)$. From where $Imz = 0$, that is $z = z_0$. □

From the above result, we can state the threshold criterion as follows:

Proposition 2. *Recalling Assumption 3. Then equilibrium (S^*, I^*, L^*) is locally asymptotically stable if $\rho(V_0) < 1$ and unstable if $\rho(V_0) > 1$.*

Proof. From Lemma 5 (items 2. and 3.), we conclude that: $\sup\{R_e z;\ 1 \in \sigma_p(V_z)\} = z_0$. Hence $s(A+C) = \sup\{R_e z;\ 1 \in \sigma_p(V_z)\} < 0$ if $\rho(V_0) < 1$, and $s(A+C) = \sup\{R_e z;\ 1 \in \sigma_p(V_z)\} > 0$ if $\rho(V_0) > 1$. □

In the following, let us note V_0^0 the operator V_0 corresponding to the case $\lambda^*(\sigma) \equiv 0$ (DFE) and V_0^* the operator V_0 corresponding to the case $\lambda^*(\sigma) > 0$ (EE). It is easily checked that

$$\chi_0^0(a,s) = \chi(a,s); \tag{52}$$

where $\chi(a,s)$ is the kernel of the Volterra operator H_0 defined by (23).

Now, the main results for the local stability of our epidemic model reads as

Theorem 3. *Let Assumptions 1 and 2 be satisfied. Let $R_0 := \rho(H_0)$ be the spectral radius of the operator H_0 defined by (22). Then,* *1) If $R_0 = \rho(H_0) < 1$ then, the unique equilibrium of (1)-(2) (DFE) is locally asymptotically stable.*

2) If $R_0 = \rho(H_0) > 1$ then, the DFE is unstable.

3) If $R_0 = \rho(H_0) > 1$ then, in addition to the DFE system (1)-(2) has at least one endemic equilibrium (EE). Moreover, if $\rho(V_0^*) < 1$ and Assumption 3 holds, then the EE is locally asymptotically stable.

Proof. For the DFE, one has $\lambda^*(\sigma) \equiv 0$. Hence, from (52) it comes that $\rho(H_0) = \rho(V_0^0) := \rho(V_0)$ (for $\lambda^* = 0$). From Prop. 2 we deduce that: if $\rho(H_0) = \rho(V_0) < 1$, the DFE is locally asymptotically stable; and unstable if $\rho(H_0) = \rho(V_0) > 1$. This end the proof of items 1. and 2.
The case of EE is a direct consequence of Prop. 2. □

Remark 1.

(♣) To emphasize the impact of vertical transmission on the spread of the disease, let us observe that the next generation operator H_0 can be rewrite as follows

$$H_0(\psi)(a) = \int_0^{a^+} \chi^{\diamondsuit}(a,s)\psi(s)ds + \int_0^{a^+} \chi_{\diamondsuit}(p,a,s)\psi(s)ds;$$

where the kernels $\chi^{\diamondsuit}(.,.)$ and $\chi_{\diamondsuit}(p,.,.)$ are

$$\begin{aligned}
\chi^{\diamondsuit}(a,s) &= \frac{S_0(s)}{l(s)} \int_s^{a^+} \beta(a,\eta)\left(\chi_{21}(\eta,s) + \chi_{31}(\eta,s)\right) d\eta; \\
\chi_{\diamondsuit}(p,a,s) &= \frac{p\chi_4(s)}{\Delta(0)} \int_0^{a^+} \beta(a,\sigma)(A_{22}(\sigma) + A_{32}(\sigma))d\sigma.
\end{aligned}$$

It is easy to see that when the proportion of infected newborns is zero ($p = 0$), then the kernel $\chi^{\diamondsuit}(0,.,.) \equiv 0$. Therefore, the vertical transmission of the disease amplifies positively the spread of the disease.

(♣♣) As a special case, we here briefly consider the proportionate mixing assumption, that is, the transmission rate β can be written as $\beta(a,s) = \beta_1(a)\beta_2(s)$ (see Dietz and Schenzle [14]; Greenhalgh,1988 [23]). In this case, the basic reproductive number $\mathcal{R}_0$ is explicitly given by:

$$R_0 := \rho(H_0) = \int_0^{a^+} \chi^{\diamondsuit}(s,s)ds + \int_0^{a^+} \chi_{\diamondsuit}(p,s,s)ds. \tag{53}$$

And the same conclusion follows as for item (♣). Thus the vertical transmission of the disease really has an impact on the dynamics and the spread of the disease into the host population. We also refer to Figures 2-4 for some illustrations of the state variables of system (1)-(2) when p takes different values: 0.02; 0.2 and 0.5.

6. Numerical analysis

In this section, we propose a numerical scheme for our model and gives some illustrations.

We adopt a finite differences scheme which is progressive of order 1 in time and regressive of order 1 in age. Our model has a structure of the following partial differential equation on the real axe:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = f(t, a). \tag{54}$$

For equation (54), the numerical scheme is defined by:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_i^n - u_{i-1}^n}{\Delta a} = f(t_n, a_i); \tag{55}$$

where i and n are the index of age and time discretization respectively; and $u_i^n := u(t_n, x_i)$.

We recall that, generally, all explicit numerical scheme is conditionally stable (Stricwerda[45]). To ensure the stability of the scheme (55) the necessary condition is the famous Courant-Friedrichs-Lewy (CFL) condition given as follow:

$$\frac{\Delta t}{\Delta a} \leqslant 1. \tag{56}$$

For a given age step discretization Δa, the restriction $\Delta t \leqslant \Delta a$ is necessary for the time step discretisation Δt.

We are able now to give the solution of the problem (1)-(2) on some time interval $[0, T]$ using the above numerical scheme.

The age-specific reproduction rate $f(a)$ is taken to be

$$f(a) = \begin{cases} \frac{1}{5}\sin^2\left(\frac{\pi(a-15)}{30}\right) \ if \ \ 15 \le a \le 45; \\ 0 \text{ if not.} \end{cases}$$

The fecundity function $f(.)$ is stated here in units of 1 / years for easier readability and assumes that from age 15 to 45 years a woman will generally give birth to three children, since $\int_0^{a^+} f(a)da = 3$, where $a^+ = 80$ is the largest age allowed for the simulation.

We also consider a low value of recruitment $\Lambda(.)$

$$\Lambda(a) = \begin{cases} \frac{1}{10}\sin^2\left(\frac{\pi(a-17)}{43}\right) \ if \ \ 17 \le a \le 60; \\ 0 \text{ if not.} \end{cases}$$

This recruitment assume that the total number of recruitment at time t is approximately equal two, that is $\int_0^{a^+} \Lambda(a) = 2.15$

The transmission coefficient $\beta(.,.)$ is assume to be

$$\beta(a, s) = \begin{cases} \beta_0 \sin^2\left(\frac{\pi(a-14)}{46}\right)\sin^2\left(\frac{\pi(s-14)}{46}\right), \text{ if } a, s \in [14, 60]; \\ 0 \text{ if not.} \end{cases}$$

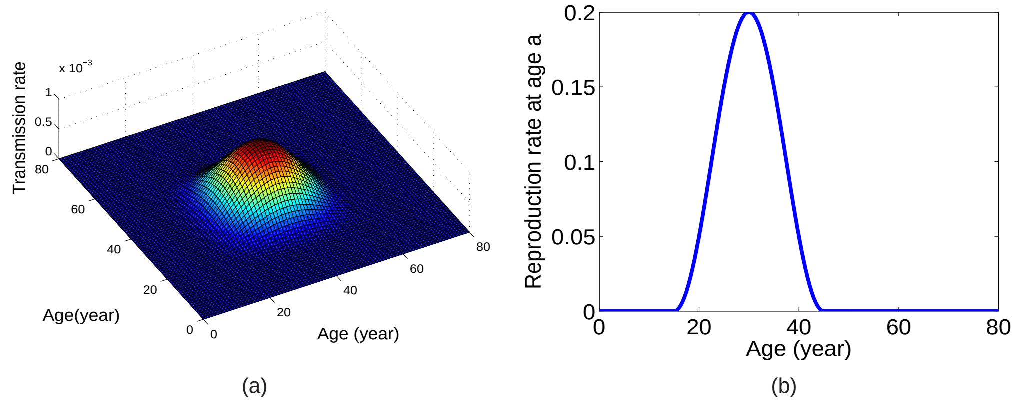

Figure 1: (1a) Transmission coefficient $\beta(.,.)$ when the transmission constant $\beta_0 = 10^{-3}$. (1b) Fecundity function $f(.)$.

Table 1: Numerical values for the parameters of the model

Parameters	Description	Estimated value
β_0	Transmission constant	Variable
p	Vertical tranmission rate	Variable
μ	Natural death rate	0.0101/yr [1]
r	Rate of effective therapy	$1/yr$ [1]
ϕ	Rate at witch infectious become loss of sight	0.75/yr [1]
γ	Rate at witch lost of sight return to the hospital	0.02/yr [1]
d_1	Death rate of infectious	0.02/yr [1]
d_2	Death rate of lost of sight	0.2/yr [1]

Note: Source of estimates.
[1] Assumed.

wherein the nonnegative constant β_0 (transmission constant) will be variable. Figure 1 illustrates the transmission coefficient β (for $\beta_0 = 10^{-3}$) and the fecundity function f. The other parameters of our system are arbitrarily chosen (see Table 1).

We provide numerical illustrations for different values of vertical transmission p: 0.02, 0.2 and 0.5

In Figure 2, the vertical transmission rate of the disease is fixed to be $p = 0.02$. We observe that infectious individuals (infected and lost of sight) are between 17 and 70 of age. The number of young infectious (namely infectious with age $a < 17$) is negligible, because the value of vertical transmission rate p is low.

In figure 3, the vertical transmission rate of the disease is fixed to be $p = 0.2$. We observe that much of the infectious individuals (infected and lost of sight) are between

17 and 70 of age. Let us also observe that the number of infectious individuals with age between 17 and 70 is approximately the same than the number of infectious individuals with age between 17 and 70 when $p = 0.02$ (see Figs 2-3). But now, there are also infectious individuals with age $a < 17$ which was not the case when $p = 0.02$.

The same observation is given by Figure 4 where the vertical transmission rate of the disease is fixed to be $p = 0.5$. Hence Figures 2-4 emphasize that the vertical transmission of the disease really has an impact on the dynamics and the spread of the disease into the host population. See also Table 2 for the impact of the vertical transmission of the disease on the spread of the epidemic.

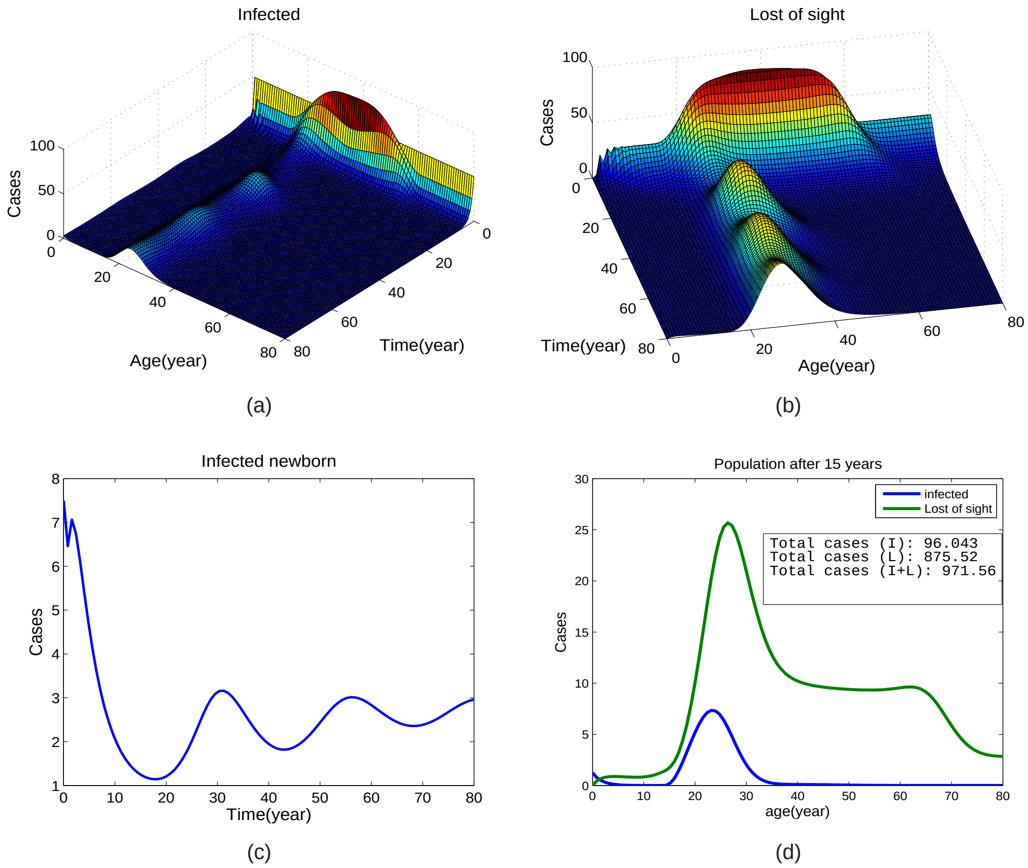

Figure 2: The transmission constant and the vertical transmission rate are fixed to be $\beta_0 = 10^{-3}$ and $p = 0.02$. The other parameters are given by Table 1. (2a) Distribution of Infected individuals. (2b) Distribution of Lost of sight. (2c) Distribution of infected newborn. (2d) Distribution of Infected and Lost of sight individuals after 80 years of time observation.

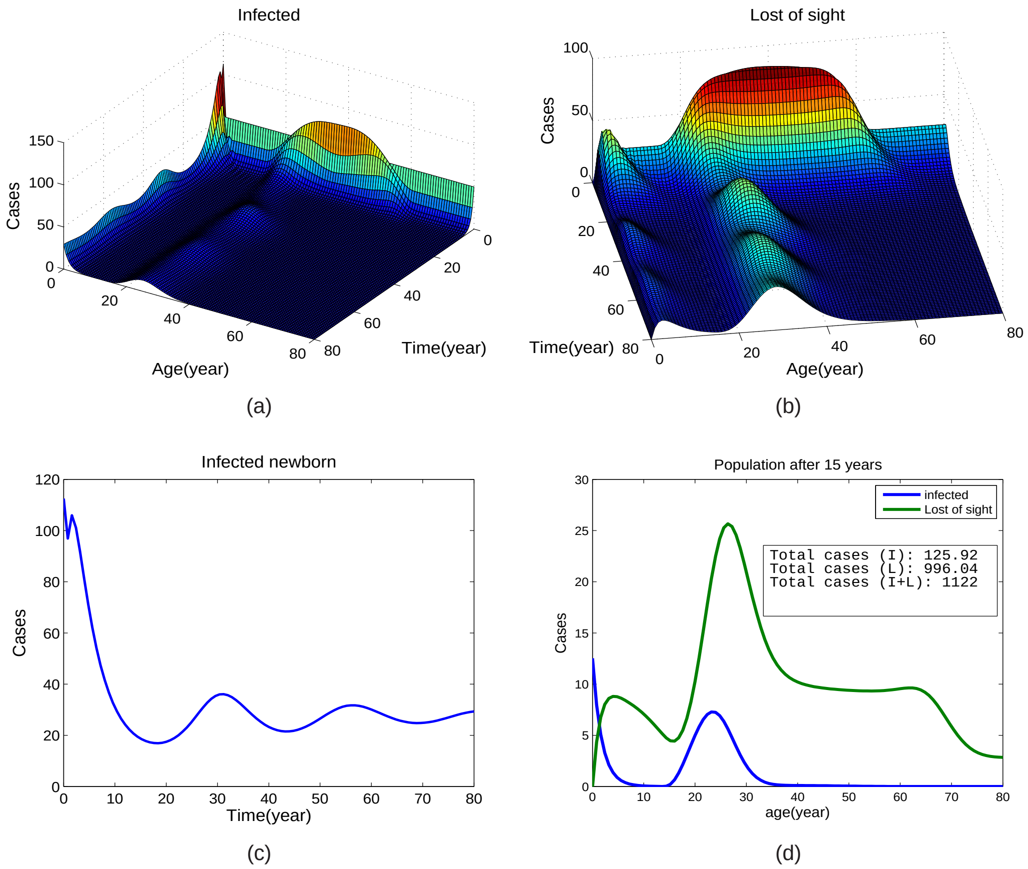

Figure 3: The transmission constant and the vertical transmission rate are fixed to be $\beta_0 = 10^{-3}$ and $p = 0.2$. The other parameters are given by Table 1. (3a) Distribution of Infected individuals. (3b) Distribution of Lost of sight. (3c) Distribution of infected newborn. (3d) Distribution of Infected and Lost of sight individuals after 80 years of time observation.

Table 2: Impact of the vertical transmission of the disease.

Vertical transmission rate (p)	Rate increase over the case when $p = 0$
$p = 0.02$	1.8%
$p = 0.2$	17.5%
$p = 0.5$	43.8%

Total cases (I+L) when $p = 0$: 954.85 cases. (i.e. when the vertical transmission of the disease is neglected in the host population.

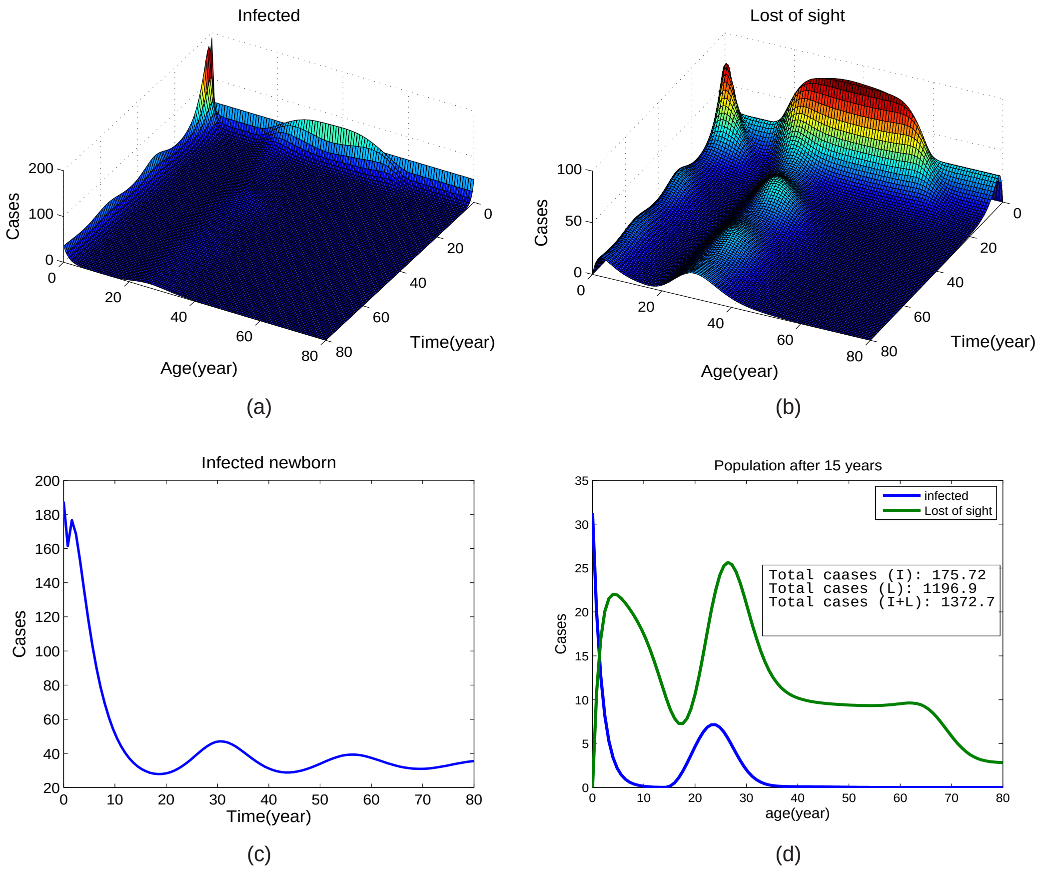

Figure 4: The transmission constant and the vertical transmission rate are fixed to be $\beta_0 = 10^{-3}$ and $p = 0.5$. The other parameters are given by Table 1. (4a) Distribution of Infected individuals. (4b) Distribution of Lost of sight. (4c) Distribution of infected newborn. (4d) Distribution of Infected and Lost of sight individuals after 80 years of time observation.

7. Conclusion

In this paper, we consider a mathematical model for the spread of a directly transmitted infections disease in an age-structured population with demographics process. The disease can be transmitted not only horizontally but also vertically from adult individuals to their children. The dynamical system is formulated with boundary conditions.

We have described the semigroup approach to the time evolution problem of the abstract epidemic system. Next we have calculated the basic reproduction ratio and proved that the disease-free steady state is locally asymptotically stable if $R_0 < 1$, and at least one endemic steady state exists if the basic reproduction ratio R_0 is greater than the unity. Moreover, we have shown that the endemic steady state is forwardly bifurcating from the disease-free steady state at $R_0 = 1$. Finally we have shown sufficient conditions which guarantee the local stability of the endemic steady state. Roughly speaking, the endemic steady state is locally asymptotically stable if it corresponds to a very small force of infection.

However the global stability of the model still an interesting open problem. Moreover, biologically appropriate assumptions for the unique existence of an endemic steady state is also not yet know.

8. References

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