Uniqueness for an hyperbolic inverse problem with time-dependent coefficient

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ABSTRACT. This paper deals with an hyperbolic inverse problem of determining a time-dependent coefficient a appearing in a dissipative wave equation, from boundary observations. We prove in dimension n greater than two, that a can be uniquely determined in a precise subset of the domain, from the knowledge of the Dirichlet-to-Neumann map.

RÉSUMÉ. Dans ce travail, on étudie le probl ème inverse de la détermination d'un coefficient dépendant de la variable d'espace et du temps apparaissant dans une équation d'onde dissipative, à partir des mesures faites sur tout le bord du domaine. On démontre que ce coefficient peut être déterminé d'une manière unique dans une partie précise du domaine à partir des mesures de type Neumann.

KEYWORDS : Inverse problems, Dissipative wave equation, Time-dependent coefficient, Uniqueness .

MOTS-CLÉS : Problèmes inverses, équation d'onde dissipative, coefficient qui dépendent du temps, Unicité

1. Introduction

1.1. Statement of the problem

The present paper is devoted to the study of the following hyperbolic inverse problem: Given $T > 0$ and a bounded domain $\Omega \subset R^n$, $n \geq 2$, with $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$, determine the absorbing coefficient a present in the following initial boundary value problem for the wave equation from boundary observations

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u = 0 & \text{in } Q = \Omega \times (0,T), \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = f(x,t) & \text{on } \Sigma = \Gamma \times (0,T), \end{cases} \tag{1}$$

where $f \in H^1(\Sigma)$, and the coefficient $a \in \mathcal{C}^2(\overline{Q})$ is assumed to be real valued. It is well known (see [10] and [9]) that if the compatibility condition is satisfied, that is $f(\cdot,0) = 0$, then, there exists a unique solution u to the equation (1) that belongs to the following space

$$u \in \mathcal{C}([0,T], H^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ such that we have

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leq \|f\|_{H^1(\Sigma)}, \tag{2}$$

where ν denotes the unit outward normal to Γ at x and $\partial_\nu u$ stands for $\nabla u \cdot \nu$. In the present paper, we focus on the uniqueness issue in the study of the inverse problem of determining the time-dependent absorbing coefficient a from the knowledge of the Dirichlet-to-Neumann map.

From a physical view point, the inverse problem under consideration consists in recovering the absorbing coefficient a in an homogeneous medium by probing it with disturbances generated on the boundary. The data are the responses of the medium to these disturbances measured on all the boundary. Here the coefficient a can be seen as one of the medium properties and we aim to recover it in a specific subset of the domain from boundary measurements, after probing the medium by a Dirichlet data f. The medium is assumed to be quiet initially.

The problem of recovering coefficients that depend only on the spatial variable is considered by many authors. In [16] Rakesh and Symes proved a uniqueness result in recovering a time-independent potential appearing in a wave equation from measurements made on the whole boundary. The main tools in the derivation of this result are first, the construction of geometric optics solutions and second, the relation linking the hyperbolic Dirichlet-to-Neumann map to the X-ray transform. As for the uniqueness from local Neumann measurements, we refer to Eskin [7]. One can also see the paper of Isakov [11], in which a uniqueness result was proved in the determination of two time-independent coefficients appearing in a dissipative wave equation.

The stability in the case where the Neumann data are observed on a subdomain of the boundary was considered by Bellassoued, Choulli and Yamamoto [1], where a stability estimate of log-type was proved in recovering a time-independent coefficient appearing in a wave equation. In [13], Isakov and Sun established a stability result of Hölder type in determining a coefficient in a subdomain from local Neumann data. As for the stability in the case where Neumann data are observed on the whole boundary, we refer to Sun [21], Cipolatti and Lopez [6].

When the coefficients depend also on the time variable, there is a uniqueness result proved by Ramm and Rakesh [17], in which they proved that a time-dependent potential appearing in a wave equation can be uniquely determined in a precise subset made of lines making an angle of $45°$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}$, from global Neumann-data. It's clear from [10] that this coefficient can not be recovered over the whole domain Q and this is actually due to the homogeneous initial conditions imposed in the system. However, Isakov proved in [12], that the time-dependent coefficient may be uniquely determined over the whole domain Q, but he needed to know much more information about the solution of the wave equation. We can also refer to [8, 18, 19, 20].

Inspired by the work of Bellassoued and Dos Santos [2], Waters [22] proved recently that one can stably recover the X-ray transform of a time-dependent lower order term present in a wave equation from the knowledge of the Dirichlet-to-Neumann map, in the Riemmanian case. In the euclidian case, Ben Aïcha [5] and Kian [14, 15], showed by taking inspiration from the work of Bellassoued-Jellali-Yamammoto [3, 4], stability results in the recovery of a zeroth order time-dependent coefficient appearing in a wave equation.

In this paper, we prove that the time-dependent absorbing coefficient a can be uniquely determined with respect to the Dirichlet-to-Neumann map in a specific subset of the domain Q, provided that a is known outside this subset.

1.2. Main results

In order to state our main result we first introduce the following notations.

Let $r > 0$ be such that $T > 2r$ and $\Omega \subseteq B(0,r/2) = \Big\{x \in R^n,\, |x| < r/2\Big\}$. We set $Q_r = B(0,r/2) \times (0,T)$. We consider the annular region around the domain Ω,

$$\mathscr{A}_r = \left\{x \in R^n,\ \frac{r}{2} < |x| < T - \frac{r}{2}\right\},$$

and the forward and backward light cones:

$$\mathscr{C}_r^+ = \left\{(x,t) \in Q_r,\ |x| < t - \frac{r}{2},\, t > \frac{r}{2}\right\},$$

$$\mathscr{C}_r^- = \left\{(x,t) \in Q_r,\ |x| < T - \frac{r}{2} - t,\, T - \frac{r}{2} > t\right\},$$

$$\mathscr{C}_r = \left\{(x,t) \in Q_r,\ |x| \leq \frac{r}{2} - t,\ 0 \leq t \leq \frac{r}{2}\right\}.$$

Finally, we denote

$$Q_r^* = \mathscr{C}_r^+ \cap \mathscr{C}_r^- \ \text{ and }\ Q_{r,*} = Q \cap Q_r^*.$$

We remark that the open subset $Q_{r,*}$ is made of lines making an angle of $45°$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}_r$. We notice that $Q_{r,*} \subset Q$. Note, that in the particular case where $\Omega = B(0,r/2)$, we have $Q_{r,*} = Q_r^*$ (see Figure 1 in [5]).

Our set of data will be given by the the Dirichlet-to-Neumann map Λ_a defined as follows

$$\begin{array}{rrcl} \Lambda_a: & H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ & f & \longmapsto & \partial_\nu u, \end{array}$$

By (2) we have that Λ_a is continuous from $H^1(\Sigma)$ to $L^2(\Sigma)$. We denote by $\|\Lambda_a\|$ its norm in $\mathcal{L}(H^1(\Sigma), L^2(\Sigma))$. Let us now introduce the admissible set of the absorbing coefficients a. Given $a_0 \in \mathcal{C}^2(\overline{Q}_r)$ and $M > 0$ we set

$$\mathcal{A}(a_0, M) = \{a \in \mathcal{C}^2(\overline{Q}_r), \quad a = a_0 \;\text{ in }\; \overline{Q}_r \setminus Q_{r,*}, \;\; \|a\|_{\mathcal{C}^2(Q)} \leq M\}.$$

Having said that we may state the main results of this paper

Theorem 1.1 *(Non uniqueness) For any $a \in \mathcal{A}(a_0, M)$ such that Supp$(a) \subset \mathscr{C}_r$, we have $\Lambda_a = \Lambda_0$.*

Theorem 1.2 *(Uniqueness) Let $T > 2$Diam(Ω) and $a_i \in \mathcal{A}(a_0, M)$, $i = 1, 2$. Then, we have*

$$\Lambda_{a_2} = \Lambda_{a_1} \quad \textit{implies} \quad a_2 = a_1 \;\; \textit{on} \;\; Q_{r,*}.$$

The outline of this paper is as follows. In Section 2, we develop the proof of Theorem 1.1. Section 3 is devoted to the construction of geometric optics solutions to the equation (1). Using these particular solutions, we prove in Section 3 Theorem 1.2.

2. Non uniqueness in determining the time-dependent coefficient

In this section we aim to show that it is hopeless to recover the time-dependent coefficient a over the whole domain in the case where the initial conditions are zero.

2.1. Preliminary

This section is devoted to the proof of a fundamental result which is borrowed from [10]. Let us first introduce the following notations. We define

$$V = \bigcup_{0 \leq \tau \leq t'} D(\tau) = \bigcup_{0 \leq \tau \leq t'} (\mathscr{C}_r \cap \{t = \tau\}),$$

where $0 < t' < r/2$. Moreover, we denote by

$$S = \partial \mathscr{C}_r \cap (\Omega \times]0, t'[), \quad \text{and} \quad \partial V = S \cup D(t') \cup D(0).$$

Lemma 2.1 *Let us denote by u the solution of the dissipative wave equation*

$$\begin{cases} \left(\partial_t^2 - \Delta + a(x,t)\partial_t\right) u(x,t) = 0 & \textit{in}\, Q, \\ u(x,0) = 0 = \partial_t u(x,0) & \textit{in}\, \Omega, \\ u(x,t) = f(x,t) & \textit{on}\, \Sigma. \end{cases}$$

Then, $u(x,t) = 0$ on the set $\mathscr{C}_r$.

We denote by $P = \partial_t^2 - \Delta + a(x,t)\partial_t$. A simple calculation gives us

$$\begin{aligned} \int_V 2Pu(x,t)\, \partial_t u(x,t)\, dx\, dt &= \int_V 2\partial_t^2 u(x,t) \partial_t u(x,t)\, dx\, dt - \int_V 2\Delta u(x,t)\, \partial_t u(x,t)\, dx\, dt \\ &\quad + \int_V 2a(x,t)|\partial_t u(x,t)|^2\, dx\, dt \\ &= \int_V \partial_t \left(|\partial_t|^2 + |\nabla u|^2\right) dx\, dt + \int_V \sum_{j=1}^n \partial_j (\partial_t u\, \partial_j u)\, dx\, dt \\ &\quad + \int_V 2a(x,t)|\partial_t u(x,t)|^2\, dx\, dt. \end{aligned}$$

Then, using the above identity, we see that

$$\int_V 2Pu(x,t)\partial_t u(x,t)\,dx\,dt = \int_V \partial_t e(x,t)\,dx\,dt + \int_V \sum_{j=1}^n \partial_j X_j(x,t)\,dx\,dt + \int_V 2a(x,t)|\partial_t u(x,t)|^2\,dx\,dt,$$

where $e(x,t) = |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2$ and $X_j(x,t) = -2\partial_t u(x,t)\partial_j u(x,t)$. Next, by applying the divergence theorem, one gets

$$\begin{aligned}\int_V 2Pu(x,t)\partial_t u(x,t)\,dx\,dt &= \int_S \Big(e(x,t)\eta + \sum_{j=1}^n X_j(x,t)\mu_j\Big)\,d\sigma + \int_{D(t')} e(x,t')\,dx - \int_{D(0)} e(x,0)\,dx \\ &\quad + \int_V 2a(x,t)|\partial_t u(x,t)|^2\,dx\,dt, \qquad (3)\end{aligned}$$

where $d\sigma$ denotes the surface element of S and the vector $(\eta,\mu_1,\mu_2,...,\mu_n) \in R^{n+1}$ is the outward unit normal vector at $(x,t) \in S$ such that

$$\eta = \left(\sum_{j=1}^n \mu_j^2\right)^{1/2}. \qquad (4)$$

On the other hand, from Cauchy-Schwarz inequality and (4), we can see that

$$\begin{aligned}\int_S (e(x,t)\eta + \sum_{j=1}^n X_j(x,t)\mu_j)\,d\sigma &\geq \int_S \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2\right)\eta - 2|\partial_t u(x,t)||\nabla u(x,t)|\,\eta\,d\sigma \\ &\geq 0. \qquad (5)\end{aligned}$$

Then, since $e(x,0) = 0$ we get from (3) and (5) this estimation

$$\int_{D(t')} e(x,t')\,dx \leq \int_V 2Pu\,\partial_t u(x,t)\,dx\,dt - \int_V 2a(x,t)|\partial_t u(x,t)|^2\,dx\,dt$$

Now, using the fact that $Pu(x,t) = 0$ for any $(x,t) \in V$, we get

$$\int_{D(t')} e(x,t')\,dx \leq C\int_0^{t'}\int_{D(t)} \left(e(x,t) + |u(x,t)|^2\right)dx\,dt, \qquad (6)$$

where, the positive constant C is depending on M. Now bearing in mind that

$$|u(x,t')|^2 = |u(x,0)|^2 + \int_0^{t'} \partial_t(|u(x,t)|^2)\,dt \leq \int_0^{t'} e(x,t)\,dx. \qquad (7)$$

Thus, from (6) and (7) we deduce that

$$\int_{D(t')} \left(e(x,t') + |u(x,t')|^2\right)dx \leq \int_0^{t'}\int_{D(t)} \left(e(x,t) + |u(x,t)|^2\right)dx\,dt.$$

In view of Gronwall's Lemma we end up deducing that $u(x,t) = 0$ for any $x \in D(t')$ and $t' \in (0, r/2)$. This completes the proof of the lemma.

2.2. Proof of Theorem 1.1

Let $a \in \mathcal{A}(a_0, M)$ such that $\mathrm{Supp}(a) \subset \mathscr{C}_r$. Let $f \in H^1(\Sigma)$ and u satisfy

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u = 0 & \text{in } Q, \\ u(x,0) = 0, \ \partial_t u(x,0) = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases}$$

Since from Lemma 2.1, we have $u = 0$ in the conic set $\mathscr{C}_r$ and using the fact that $\mathrm{Supp}(a) \subset \mathscr{C}_r$, we deduce that u solves also the following hyperbolic boundary-value problem

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in } Q, \\ v(x,0) = 0, \ \partial_t v(x,0) = 0 & \text{in } \Omega, \\ v = f & \text{on } \Sigma. \end{cases}$$

Then, we conclude that $\Lambda_a(f) = \Lambda_0(f)$ for all $f \in H^1(\Sigma)$.

3. Construction of geometric optics solutions

In this section, we construct suitable geometrical optics solutions for the dissipative wave equation (1), which are key ingredients to the proof of our main result. We first state the following lemma that will be used in order to prove the main statement of this section.

Lemma 3.1 *(see [10]) Let T, M_1, $M_2 > 0$, $a \in L^\infty(Q)$ and $b \in L^\infty(Q)$, such that $\|a\|_{L^\infty(Q)} \leq M_1$ and $\|b\|_{L^\infty(Q)} \leq M_2$. Assume that $F \in L^1(0,T;L^2(\Omega))$. Then, there exists a unique solution u to the following equation*

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u(x,t) = F(x,t) & \textit{in } Q, \\ u(x,0) = 0 = \partial_t u(x,0) & \textit{in } \Omega, \\ u(x,t) = 0 & \textit{on } \Sigma, \end{cases} \tag{8}$$

such that

$$u \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ such that for $t \in (0,T)$ we have

$$\|\partial_t u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \leq C\|F\|_{L^1(0,T;L^2(\Omega))}. \tag{9}$$

By the use of the above lemma, we may construct suitable geometrical optics solutions to the equation (1) and to the retrograde problem. We shall first consider a function $\varphi \in \mathcal{C}_0^\infty(R^n)$. Notice that for all $\omega \in S^{n-1}$ the function ϕ given by

$$\phi(x,t) = \varphi(x + t\omega), \tag{10}$$

solves the following transport equation

$$(\partial_t - \omega \cdot \nabla)\phi(x,t) = 0. \tag{11}$$

Let us now prove the following Lemma.

Lemma 3.2 *Let M_1, $M_2 > 0$, $a \in \mathcal{A}(a_0, M_1)$ and $b \in W^{1,\infty}(Q)$ such that $\|b\|_{W^{1,\infty}(Q)} \leq M_2$. Given $\omega \in S^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(R^n)$, we consider the function ϕ defined by (10). Then, for any $\lambda > 0$, the following equation*

$$\partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 \quad \text{in } Q, \tag{12}$$

admits a unique solution

$$u^+ \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

of the following form

$$u^+(x,t) = \phi(x,t)\widetilde{a}^+(x,t)e^{i\lambda(x\cdot\omega+t)} + r^+(x,t), \tag{13}$$

where $\widetilde{a}^+(x,t)$ is given by

$$\widetilde{a}^+(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big), \tag{14}$$

and $r^+(x,t)$ satisfies

$$r^+(x,0) = \partial_t r^+(x,0) = 0, \ \text{in } \Omega, \quad r^+(x,t) = 0 \ \text{on } \Sigma. \tag{15}$$

Moreover, there exists a positive constant $C > 0$ such that

$$\lambda\|r^+\|_{L^2(Q)} + \|\partial_t r^+\|_{L^2(Q)} \leq C\|\varphi\|_{H^3(R^n)}. \tag{16}$$

In order to prove this lemma, it will be enough to prove the existence of r^+ satisfying

$$\begin{cases} \Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)r^+ = g(x,t), \\ r^+(x,0) = \partial_t r^+(x,0) = 0, \\ r^+(x,t) = 0, \end{cases} \tag{17}$$

and obeying the estimate (16), where $g(x,t)$ is given by

$$g(x,t) = -\Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)\widetilde{a}^+(x,t)e^{i\lambda(x\cdot\omega+t)}\Big).$$

Bearing in mind that $\widetilde{a}^+(x,t)$ solves the following equation

$$2(\partial_t - \omega\cdot\nabla)\widetilde{a}^+(x,t) = -a(x,t)\widetilde{a}^+(x,t),$$

one can see from (11) that

$$g(x,t) = -e^{i\lambda(x\cdot\omega+t)}\Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)\widetilde{a}^+(x,t)\Big) = -e^{i\lambda(x\cdot\omega+t)}g_0(x,t),$$

where $g_0 \in L^1(0,T,L^2(\Omega))$. Hence, in light of Lemma 3.1, we deduce the existence of a unique solution r^+ such that

$$r^+ \in \mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

and satisfying (17). We set

$$w(x,t) = \int_0^t r^+(x,s)\,ds. \tag{18}$$

Then, in light of (17) and (18), we get

$$\begin{aligned}\Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)w(x,t) &= \int_0^t g(x,s)\,ds + \int_0^t \Big(b(x,t) - b(x,s)\Big)r^+(x,s)\,ds \\ &\quad + \int_0^t \partial_s a(x,s) r^+(x,s)\,ds,\end{aligned}$$

Then, the function w is a solution to the following equation

$$\begin{cases} \Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)w(x,t) = F_1(x,t) + F_2(x,t) & \text{in } Q, \\ w(x,0) = 0 = \partial_t w(x,0) & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \Sigma. \end{cases}$$

Here F_1 and F_2 are given by

$$F_1(x,t) = \int_0^t g(x,s)\,ds, \tag{19}$$

and

$$F_2(x,t) = \int_0^t \Big(b(x,t) - b(x,s)\Big)r^+(x,s)\,ds + \int_0^t \partial_s a(x,s) r^+(x,s)\,ds.$$

Considering $\tau \in [0,T]$ and applying Lemma 3.1 on the interval $[0,\tau]$, we obtain

$$\|\partial_t w(.,\tau)\|^2_{L^2(\Omega)} \leq C\Big(\|F_1\|^2_{L^2(Q)} + T\left(M_1^2 + 4M_2^2\right)\int_0^\tau \int_\Omega \int_0^t |r^+(x,s)|^2\,ds\,dx\,dt\Big).$$

Therefore, in view of (18), we deduce that

$$\begin{aligned}\|\partial_t w(.,\tau)\|^2_{L^2(\Omega)} &\leq C\Big(\|F_1\|^2_{L^2(Q)} + \int_0^\tau \int_0^t \|\partial_s w(.,s)\|^2_{L^2(\Omega)}\,ds\,dt\Big) \\ &\leq C\Big(\|F_1\|^2_{L^2(Q)} + T\int_0^\tau \|\partial_s w(.,s)\|^2_{L^2(\Omega)}\,ds\Big).\end{aligned}$$

As a consequence, we find out from Gronwall's Lemma that

$$\|\partial_t w(.,\tau)\|^2_{L^2(\Omega)} \leq C\|F_1\|^2_{L^2(Q)}.$$

Hence, from (18), one deduce that $\|r^+\|_{L^2(Q)} \leq C\|F_1\|_{L^2(Q)}$. In view of (19), one can easily see that F_1 can be written as follows

$$F_1(x,t) = \int_0^t g(x,s)\,ds = \frac{1}{i\lambda}\int_0^t g_0(x,s)\partial_s(e^{i\lambda(x\cdot\omega+s)})\,ds.$$

Therefore, by integrating by parts with respect to s, we get

$$\|r^+\|_{L^2(Q)} \leq \frac{C}{\lambda}\|\varphi\|_{H^3(R^n)},$$

for some $C > 0$. Since $\|g\|_{L^2(Q)} \leq C\|\varphi\|_{H^3(R^n)}$ and using the energy estimate (9) associated to the problem (17) we obtain the following estimation

$$\|\partial_t r^+\|_{L^2(Q)} + \|\nabla r^+\|_{L^2(Q)} \leq C\|\varphi\|_{H^3(R^n)}.$$

This completes the proof of the lemma.

Lemma 3.3 *Let $M_1,\ M_2 > 0$, $a \in \mathcal{A}(a_0, M_1)$, and $b \in W^{1,\infty}(Q)$ such that $\|b\|_{W^{1,\infty}(Q)} \le M_2$. Given $\omega \in S^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(R^n)$, we consider the function ϕ defined by (10). Then, the following equation*

$$\partial_t^2 u - \Delta u - a(x,t)\partial_t u + b(x,t)u = 0 \quad \textit{in } Q, \tag{20}$$

admits a unique solution

$$u^- \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

of the following form

$$u^-(x,t) = \varphi(x + t\omega)\widetilde{a}^-(x,t)e^{-i\lambda(x\cdot\omega+t)} + r^-(x,t), \tag{21}$$

where $\widetilde{a}^-(x,t)$ is given by

$$\widetilde{a}^-(x,t) = \exp\Big(\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big), \tag{22}$$

and $r^-(x,t)$ satisfies

$$r^-(x,T) = \partial_t r^-(x,T) = 0, \ \textit{in } \Omega, \quad r^-(x,t) = 0 \ \textit{on } \Sigma. \tag{23}$$

Moreover, there exists a constant $C > 0$ such that

$$\lambda\|r^-\|_{L^2(Q)} + \|\partial_t r^-\|_{L^2(Q)} \le C\|\varphi\|_{H^3(R^n)}. \tag{24}$$

We prove this result by proceeding as in the proof of Lemma 3.2. Putting

$$\widetilde{g}(x,t) = -\Big(\partial_t^2 - \Delta - a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)\widetilde{a}^-(x,t)e^{-i\lambda(x\cdot\omega+t)}\Big).$$

Then, it is easy to see that if $r^-(x,t)$ is solution to the following system

$$\begin{cases} \Big(\partial_t^2 - \Delta - a(x,t)\partial_t + b(x,t)\Big)r^-(x,t) = \widetilde{g}(x,t) & \text{in } Q, \\ r^-(x,T) = 0 = \partial_t r^-(x,T) & \text{in } \Omega, \\ r^-(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

then, $r^+(x,t) = r^-(x, T-t)$ is a solution to (17) with $g(x,t) = \widetilde{g}(x, T-t)$ and $a(x,t)$, $b(x,t)$ are replaced by $a(x, T-t)$ and $= b(x, T-t)$.

4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The proof is based on the geometric optics solutions constructed in Section 3 and the following preliminary identity. We need first to introduce the following notations. Let $\omega \in S^{n-1}$, $a_1,\ a_2 \in \mathcal{A}(a_0, M)$. We set

$$\widetilde{a}(x,t) = (\widetilde{a}^-\widetilde{a}^+)(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big),$$

where $\widetilde{a}_1^-$ and $\widetilde{a}_2^+$ are given by

$$\widetilde{a}_1^-(x,t) = \exp\Big(\frac{1}{2}\int_0^t a_1(x+(t-s)\omega, s)\,ds\Big), \ \ \widetilde{a}_2^+(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a_2(x+(t-s)\omega, s)\,ds\Big).$$

Moreover, we define a in R^{n+1} by $a = a_2 - a_1$ in $\overline{Q}_r$ and $a = 0$ on $R^{n+1} \smallsetminus \overline{Q}_r$.

4.1. An identity for the absorbing coefficient

The main purpose of this section is to give a preliminary identity for the absorbing coefficient a .

Lemma 4.1 *Let $\varphi \in \mathcal{C}_0^\infty(\mathcal{A}_r)$ and $a_1, a_2 \in \mathcal{A}(a_0, M)$. Assume that $\Lambda_{a_2} = \Lambda_{a_1}$, then, the following identity holds*

$$\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}(x,t)\,dx\,dt = 0. \tag{25}$$

In light of Lemma 3.2, there exists a geometrical optics solution u^+ to the equation

$$\begin{cases} \partial_t^2 u^+ - \Delta u^+ + a_2(x,t)\partial_t u^+ = 0 & \text{in } Q, \\ u^+(x,0) = \partial_t u^+(x,0) = 0 & \text{in } \Omega, \end{cases}$$

in the following form

$$u^+(x,t) = \varphi(x+t\omega)\widetilde{a}_2^+(x,t)e^{i\lambda(x\cdot\omega+t)} + r^+(x,t), \tag{26}$$

corresponding to the coefficients a_2, where $r^+(x,t)$ satisfies (15), (16). We denote by f_λ the function

$$f_\lambda(x,t) = u^+(x,t)_{|\Sigma} = \varphi(x+t\omega)\widetilde{a}_2^+(x,t)e^{i\lambda(x\cdot\omega+t)}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x,t)\partial_t u_1 = 0 & \text{in } Q, \\ u_1(x,0) = \partial_t u_1(x,0) = 0 & \text{in } \Omega, \\ u_1(x,t) = f_\lambda(x,t) & \text{on } \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a_1(x,t)\partial_t u = a(x,t)\partial_t u^+ & \text{in } Q, \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \Sigma. \end{cases} \tag{27}$$

where $a = a_2 - a_1$. On the other hand Lemma 3.3 guarantees the existence of a geometrical optic solution u^- to the adjoint problem of (1)

$$\begin{cases} \partial_t^2 u^- - \Delta u^- - a_1(x,t)\partial_t u^- - \partial_t a_1(x,t)u^- = 0 & \text{in } Q, \\ u^-(x,T) = \partial_t u^-(x,T) = 0 & \text{in } \Omega, \end{cases}$$

corresponding to the coefficients a_1 and $-\partial_t a_1$, in the form

$$u^-(x,t) = \varphi(x+t\omega)e^{-i\lambda(x\cdot\omega+t)}\widetilde{a}_1{}^-(x,t) + r^-(x,t), \tag{28}$$

where $r^-(x,t)$ satisfies (23), (24). Multiplying the first equation of (27) by u^-, integrating by parts and using Green's formula, we obtain

$$\int_Q a(x,t)\partial_t u^+\, u^-\,dx\,dt = \int_\Sigma (\Lambda_{a_2} - \Lambda_{a_1})(f_\lambda)\, u^-\,d\sigma\,dt. \tag{29}$$

On the other hand, by replacing u^+ and u^- by their expressions, we get

$$\begin{aligned}&\int_Q a(x,t)\partial_t u^+ u^- \,dx\,dt = \int_Q a(x,t)\partial_t\varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\widetilde{a}_2^+ r^- \,dx\,dt\\&+\int_Q a(x,t)\varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\partial_t\widetilde{a}_2^+ r^- \,dx\,dt+\int_Q a(x,t)\partial_t\varphi(x+t\omega)\varphi(x+t\omega)(\widetilde{a}_2^+\widetilde{a}_1^-)dx\,dt\\&\quad+\int_Q a(x,t)\varphi^2(x+t\omega)\partial_t\widetilde{a}_2^+\widetilde{a}_1^- \,dx\,dt+ i\lambda\int_0^T\int_\Omega a(x,t)\varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\widetilde{a}_2^+ r^- \,dx\,dt\\&\quad+\int_Q a(x,t)\varphi(x+t\omega)e^{-i\lambda(x\cdot\omega+t)}\widetilde{a}_1^-\partial_t r^+ dx\,dt+i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)(\widetilde{a}_2^+\widetilde{a}_1^-)\,dx\,dt\\&\quad+\int_Q a(x,t)\partial_t r^+ r^- \,dx\,dt = i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}\,dx\,dt+\mathcal{I}(\lambda),\end{aligned}$$

where $\widetilde{a}=\widetilde{a}_2^+\widetilde{a}_1^-$. Then, in light of (29), we have

$$i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}(x,t)\,dx\,dt=\int_\Sigma(\Lambda_{a_2}-\Lambda_{a_1})(f_\lambda)\,u^-\,d\sigma\,dt-\mathcal{I}(\lambda). \tag{30}$$

Note that for λ sufficiently large, we have

$$|\mathcal{I}_\lambda|\le C\|\varphi\|^2_{H^3(R^n)}. \tag{31}$$

Hence, using the fact that $\Lambda_{a_2}=\Lambda_{a_1}$, we deduce from (30), (31) and by taking $\lambda\to+\infty$ the desired result.

4.2. End of the proof

In this section we complete the proof of Theorem 1.2 by the use of the results we have already obtained in the previous sections. Let us first consider the following set

$$E=\{(\xi,\tau)\in R^n\smallsetminus\{O_{R^n}\}\times R,\ |\tau|<|\xi|\},$$

and denote by $\widehat{F}$ the Fourier transform of $F\in L^1(R^{n+1})$ as follows:

$$\widehat{F}(\xi,\tau)=\int_R\int_{R^n}F(x,t)e^{-ix\cdot\xi}e^{-it\tau}\,dx\,dt.$$

In light of (25), we have as λ goes to $+\infty$, the following identity

$$\int_Q a(x,t)\varphi^2(x+t\omega)\exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega,s)\,ds\Big)\,dx\,dt=0. \tag{32}$$

Then, using the fact $a(x,t)=0$ outside $Q_{r,*}$ and making this change of variables $y=x+t\omega$, one gets

$$\int_0^T\int_{R^n}a(y-t\omega,t)\varphi^2(y)\exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\,dy\,dt=0.$$

Bearing in mind that

$$\begin{aligned}\int_0^T\int_{R^n}\ \ a(y-t\omega,t)\ \ &\varphi^2(y)\exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\,dy\,dt\\&= -2\int_0^T\int_{R^n}\varphi^2(y)\frac{d}{dt}\Big[\exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\Big]\,dy\,dt\\&= -2\int_{R^n}\varphi^2(y)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dy.\end{aligned}$$

we conclude that

$$\int_{R^n} \varphi^2(y)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dy = 0. \tag{33}$$

Now, we consider a nonnegative function $\psi \in \mathcal{C}_0^\infty(R^n)$ supported in the unit ball $B(0,1)$ such that $\|\psi\|_{L^2(R^n)} = 1$. We define

$$\varphi_h(x) = h^{-n/2}\psi\Big(\frac{x-y}{h}\Big), \tag{34}$$

where $y \in \mathcal{A}_r$. Then, for $h>0$ sufficiently small one can see that Supp $\varphi_h \subset \mathcal{C}_0^\infty(\mathcal{A}_r)$ and satisfies

$$\text{Supp}\,\varphi_h \cap \Omega = \varnothing, \quad \text{and} \quad \text{Supp}\,\varphi_h \pm T\omega \cap \Omega = \varnothing.$$

Then, as h goes to 0 we deduce from (33) with $\varphi = \varphi_h$ that

$$\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1 = 0.$$

Since $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, we conclude that

$$\int_R a(y-t\omega,t)\,dt = 0, \quad \text{a.e } \; y \in \mathcal{A}_r, \; \omega \in S^{n-1}. \tag{35}$$

On the other hand, if $|y| \leq \dfrac{r}{2}$, we notice that

$$a(y-t\omega,t) = 0, \;\; \forall\, t \in R. \tag{36}$$

Indeed, we have

$$|y-t\omega| \geq |t| - |y| \geq t - \frac{r}{2}, \tag{37}$$

hence, $(y-t\omega,t) \notin \mathscr{C}_r^+$ if $t > r/2$, from (37). As $(y-t\omega,t) \notin \mathscr{C}_r^+$ if $t \leq r/2$, then we have $(y-t\omega,t) \notin \mathscr{C}_r^+ \supset Q_{r,*}$ for $t \in R$. This and the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, yield (36), and consequently,

$$\int_R a(y-t\omega,t)\,dt = 0, \quad |y| \leq \frac{r}{2}.$$

By a similar way, we prove for $|y| \geq T - r/2$, that $(y-t\omega,t) \notin \mathscr{C}_r^- \supset Q_{r,*}$, for $t \in R$. Then we obtain

$$\int_R a(y-t\omega,t)\,dt = 0, \;\; \text{a.e. } y \notin \mathscr{A}_r, \;\; \omega \in S^{n-1}. \tag{38}$$

Thus, by (35) and (38) we find

$$\int_R a(y-t\omega,t)\,dt = 0, \;\; \text{a.e } \; y \in R^n, \; \omega \in S^{n-1}.$$

We now turn our attention to the fourier transform of a. Let $\xi \in R^n$. In light of (38) and by the use of Fubini's Theorem, we get

$$\int_R \int_{R^n} a(x-t\omega,t)e^{-ix\cdot\xi}\,dx\,dt = 0.$$

Making the change of variables $y = x - t\omega$, one gets

$$\int_R \int_{R^n} a(y,t)e^{-iy\cdot\xi}e^{-it(\omega\cdot\xi)}\,dy\,dt = 0.$$

Let us now consider $\xi' \in S^{n-1}$ such that $\xi \cdot \xi' = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \xi' \in S^{n-1},$$

then $(\xi, \tau) = (\xi, \omega \cdot \xi) \in E$. We then deduce that $\widehat{a}(\xi, \tau) = 0$ in the set E. By an argument of analyticity, we extend this result to R^{n+1}. Hence, by the injectivity of the Fourier transform we get the desired result. This completes the proof of Theorem 1.2.

5. References

[1] M. BELLASSOUED, , M. CHOULLI, , M. YAMAMOTO, Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem, *J. Diff. Equat.*, vol. 247, 2,num. 465-494, 2009.

[2] M. BELLASSOUED, , D. DOS SANTOS FERREIRA, Stability estimates for the anisotripic wave equation from the Dirichlet-to-Neumann map , *Inverse Probl. Imaging*, 5, 4, 745-73, 2011.

[3] M. BELLASSOUED, , D. JELLALI ,, M. YAMAMOTO, Lipschitz stability for a hyperbolic inverse problem by finite local boundary data , *Applicable Analysis*, 85, 1219-1243, 2006.

[4] M. BELLASSOUED, , D. JELLALI, , M. YAMAMOTO, Stability estimate for the hyperbolic inverse boundary value problem by local Dirichlet-to-Neumann map , *J. Math. Anal. Appl.*, 343, 2, 1036-1046, 2008.

[5] I. BEN AÏCHA IBTISSEM , Stability estimate for a hyperbolic inverse problem with time-dependent coefficient , *Inverse Problems*, 31, 125010, 21pp, 2015.

[6] R. CIPOLATTI, , IVO F. LOPEZ , Determination of coefficients for a dissipative wave equation via boundary measurements , *J. Math. Anal. Appl.*, 306, 317-329, 2005.

[7] G. ESKIN , A new approach to hyperbolic inverse problems , *Inverse problems*, 22 no. 3, 815-831, 2006.

[8] G. ESKIN , Inverse hyperbolic problems with time-dependent coefficients , *Commun. Partial Diff. Eqns.*, 32, 11, 1737-1758, 2007.

[9] M. IKAWA , A mixed problem for hyperbolic equations of second order with a first order derivative boundary condition , *Publ.RIMS, Kyoto Univ*,1969.

[10] M. IKAWA , Hyperbolic Partial Differential Equations and Wave Phenomena , *Providence, RI American Mathematical Soc.*, 2000.

[11] V. ISAKOV, An inverse hyperblic problem with many boundary measurements , *Commun. Partial Diff. Eqns.*, 16, 1183-1195, 1991.

[12] V. ISAKOV, Completeness of products of solutions and some inverse problems for PDE , *J.Diff. Equat.*, 92, 305-316, 1991.

[13] V. ISAKOV, , Z. SUN, Stability estimates for hyperbolic inverse problems with local boundary data , *Inverse problems,* 8, 193-206, 1992.

[14] Y. KIAN , " Unique determination of a time-dependent potential for wave equations from partial data, *preprint, arXiv:1505.06498.*

[15] Y. KIAN, " Stability in the determination of a time-dependent coefficient for wave equations from partial data , *arXiv:1406.5734.*

[16] RAKESH, , W. SYMES," Uniqueness for an inverse problem for the wave equation , *Comm. in PDE*, 13, 1, 87-96, 1988.

[17] A. G. RAMM, , RAKESH, " Property C and an Inverse Problem for a Hyperbolic Equation , *J.Math. Anal. Appl.*, 156, 209-219, 1991.

[18] A.G. RAMM, , SJÖSTRAND, An inverse problem of the wave equation , *Math. Z.*, 206, 119-130, 1991.

[19] R. SALAZAR , Determination of time-dependent coefficients for a hyperbolic inverse problem , *Inverse Problems*, 29, 9, 095015, 2013.

[20] P. STEFANOV, Uniqueness of the multi-dimentionnal inverse scattering problem for time-dependent potentials, *Math. Z.*, 201, 4, 541-559, 1994.

[21] Z. SUN, On continuous dependence for an inverse initial boundary value problem for the wave equation , *J. Math. Anal. Appl.*, 150, 188-204, 1990.

[22] A. Waters, Stable determination of X-ray transforms of time-dependent potentials from partial boundary data , *Commun. Partial Diff. Eqns.*, 39, 2169-2197, 2014.