Inverse Problem:

Stability for the aligned magnetic field by the Dirichlet-to-Neumann map for the wave equation in a periodic quantum waveguide

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RÉSUMÉ. Dans ce papier, on a prouvé une estimation de stabilité pour le problème inverse de détermination du champ magnétique dans l'équation des ondes donné sur un domaine non borné à partir de l'opérateur de Dirichlet-to-Neumann. On a montré un résultat de stabilité pour ce problème inverse, dont la démonstration est basée sur la construction de solutions optique géométrique pour l'équation des ondes avec un potentiel magnétique 1-périodique.

ABSTRACT. We consider the boundary inverse problem of determining the aligned magnetic field appearing in the magnetic wave equation in a periodic quantum cylindrical waveguide from boundary observations. The observation is given by the Dirichlet to Neumann map associated to the wave equation. We prove by means of the geometrical optics solutions of the magnetic wave equation that the knowledge of the Dirichlet-to-Neumann map determines uniquely the aligned magnetic field induced by a time independent and 1-periodic magnetic potential. We establish a Hölder-type stability estimate in the inverse problem.

MOTS-CLÉS : Problème inverse, l'équation des ondes, l'opérateur de Dirichlet-to-Neumann

KEYWORDS : Inverse problem, Magnetic wave equation, Dirichlet-to-Neumann map

1. Introduction

Let $\Omega = \mathbb{R} \times \omega$ be an infinite waveguide, where ω is a bounded domain of $\mathbb{R}^2$, with C^2-boundary $\partial\omega$. Throughout this text we write $x = (x_1, x')$ with $x' = (x_2, x_3)$ for every $x = (x_1, x_2, x_3) \in \Omega$. Let $A = (a_j)_{1\leq j\leq 3} \in W^{3,\infty}(\Omega; \mathbb{R}^3)$ be time independent and 1-periodic magnetic potential with respect to x_1 i.e.

$$A(x_1 + 1, x') = A(x_1, x'),\ (x_1, x') \in \Omega = \mathbb{R} \times \omega. \tag{1.1}$$

Given $T > 0$, we consider the initial boundary value problem (IBVP) for the wave equation,

$$\begin{cases} (\partial_t^2 - \Delta_A)u = 0 & \text{in } Q = (0,T) \times \Omega, \\ u(0,\cdot) = 0 & \text{in } \Omega, \\ \partial_t u(0,\cdot) = 0 & \text{in } \Omega \\ u = g & \text{on } \Sigma = (0,T) \times \partial\Omega, \end{cases} \tag{1.2}$$

where Δ_A is the magnetic Laplacian defined by

$$\Delta_A = \sum_{j=1}^{3} (\partial_j + ia_j)^2 = \Delta + 2iA\cdot\nabla + i\mathrm{div}(A) - |A|^2.$$

In this paper, we are interested in determining the magnetic potential A from the knowledge of the Dirichlet-to-Neumann (abbreviated to DN in the following) map associated with A

$$\Lambda_A(g) = (\partial_\nu + iA\cdot\nu)\,u, \tag{1.3}$$

where u is the solution to (1.2) and $\nu = \nu\,(x)$ denotes the unit outward normal to $\partial\Omega$ at x. As was noted in [8], the DN map is invariant under the gauge transformation of the magnetic potential : Namely, given $\Psi \in \mathcal{C}^1(\Omega)$ such that $\Psi_{|\partial\Omega} = 0$, it ensues from the identities,

$$e^{-i\Psi}\Delta_A e^{i\Psi} = \Delta_{A+\nabla\Psi}, \quad e^{-i\Psi}\Lambda_A e^{i\Psi} = \Lambda_{A+\nabla\Psi}, \tag{1.4}$$

that

$$\Lambda_A = \Lambda_{A+\nabla\Psi}.$$

From this information and from a geometric view point, we may reformulate the basic inverse problem considered in this article as follows.
Inverse problem for the magnetic wave equation : is it possible to determine the magnetic field $d\alpha_A$ given by

$$d\alpha_A = \sum_{i,j=1}^{3} (\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i})\,\mathrm{d}x_j \wedge \mathrm{d}x_i,$$

from the knowledge of the boundary measurements Λ_A.
The problem inverse of recovering time-independent coefficient for partial differential equations such as the wave equation have attracted many attention over the recent years. For example, in [14], Rakesh study the determination of the time-independent scalar potentials in a wave equations, from the DN map. Bellassoued and Benjoud proved in [2] that the knowledge of the Dirichlet-to-Neumann map for the magnetic wave equation measured on the boundary determines uniquely the magnetic field. There methods is essential based on the construction of geometric optics solution.

In this results, the DN map gives on the whole boundary. The uniqueness by a local DN map is well solved (Belishev [1], Eskin-Ralston [9], Eskin [6], Katchalov, Kurylev, and Lassas [11]).
Note here that all these results are concerned in a bounded domain. In this article We consider the inverse problem of determining the magnetic field appearing in the magnetic wave equation in an infinite cylindrical domain. There are only a small number of mathematical papers dealing with with inverse boundary problems in an unbounded domain. In [13], Li and Uhlmann proved that the knowledge of DN map determines uniquely the scalar potential in an infinite slab. In [5] Choulli , Kian, and Socoorsi proved a logarithmic stability in the determination of the time-dependent scalar potential in a 1-periodic quantum cylindrical waveguide, from the boundary measurements of the solution to the dynamic Schrödinger equation. See also the refs. ([10], [12] and [13]).

1.1. Notations

Throughout this text we denote by

$$\tilde{\Omega} = (0,1)\times\omega,\ \tilde{Q} = (0,T)\times\tilde{\Omega},\ \tilde{\Sigma} = (0,T)\times(0,1)\times\partial\omega. \tag{1.5}$$

Further, we denote by $|y| := \left(\sum_{i=1}^{3} y_i^2\right)^{1/2}$ the Euclidian norm of $y=(y_1,y_2,y_3)\in\mathbb{R}^3$ and we write

$$B(x_0,r) = \{x=(x_1,x_2,x_3)\in\mathbb{R}^3;\quad |x-x_0|\le r\},\quad \text{for all } r>0.$$

We note $H^p(\Omega)$ the p-th order Sobolev space on Ω for every $p\in\mathbb{N}$, where $H^0(\Omega)$ stands for $L^2(\Omega)$. Finally, we put

$$D_\omega = \inf\{R\in\mathbb{R}^+ :\ \omega\subset B(x_0',R)\ \text{ for some }\ x_0'\in\mathbb{R}^2\}.$$

We may now define the trace operator τ by

$$\tau w = w_{|\Sigma},\ \text{ for } w\in C_0^\infty([0,T]\times\mathbb{R}, C^\infty(\overline{\omega})).$$

Recall that since ω is a bounded domain of $\mathbb{R}^2$ with C^2-boundary, we can extend τ to a bounded operator from $H^2\left(0,T;H^2(\Omega)\right)$ into $L^2\left((0,T)\times\mathbb{R},H^{3/2}(\partial\omega)\right)$. Then the space $X_0=\tau H^2\left(0,T;H^2(\Omega)\right)$ endowed with the norm

$$\|\ w\ \|_{X_0} = \inf\{\|W\|_{H^2(0,T;H^2(\Omega))};\ W\in H^2\left(0,T;H^2(\Omega)\right)\ \text{ such that } \tau W = w\},$$

is Hilbertian. Moreover, the linear operator Λ_A defined by (1.3), is bounded from X_0 to $L^2(\Sigma)$.

1.2. Main results

In this subsection we state the main results of this article. Ours first result can be stated as follows.

Theorem 1.1 *We consider two potentials $A_i,\ i=1,2$ in $W^{3,\infty}(\Omega;\mathbb{R}^3)$ obey the conditions $\|A_i\|_{W^{3,\infty}(\tilde{\Omega})}\le M,\ \ i=1,2$, For $M>0$ be fixed. Assume moreover that*

$$A_1 = A_2\ \ on\ \ (0,1)\times\partial\omega, \tag{1.6}$$

$$\partial_j A_1 = \partial_j A_2\ \ on\ \ (0,1)\times\partial\omega,\ \ j=1,2,3. \tag{1.7}$$

Then there exist a constant $C > 0$ and $\mu \in (0,1)$, such that

$$\left\| \frac{\partial a_2}{\partial x_3} - \frac{\partial a_3}{\partial x_2} \right\|_{H^{-1}(\tilde{\Omega})} \leq C \|\Lambda_{A_2} - \Lambda_{A_1}\|^{\mu}.$$

where C depends on T, ω and M.

Theorem 1.1 follows from a result we shall make precise below, which is related to the following IBVP with quasi-periodic boundary conditions,

$$\begin{cases} (\partial_t^2 - \Delta_A)u = 0 & \text{in } \tilde{Q}, \\ u(0,\cdot) = 0 & \text{in } \tilde{\Omega}, \\ \partial_t u(0,\cdot) = 0 & \text{in } \tilde{\Omega}, \\ u = h & \text{on } \tilde{\Sigma}, \\ u(\cdot,1,\cdot) = e^{i\theta} u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \\ \partial_{x_1} u(\cdot,1,\cdot) = e^{i\theta}\partial_{x_1} u(\cdot,0,\cdot) & \text{on } (0,T)\times\omega, \end{cases} \tag{1.8}$$

where θ is arbitrarily fixed in $[0,2\pi)$. To this purpose, for any subspace $\mathcal{R} = (0,1)\times\mathbb{R}^2$ or $\mathbb{R}^3$, we take

$$H^1_\theta(\mathcal{R}) = \{u \in H^1(\mathcal{R});\ u(1,\cdot) = e^{i\theta}u(0,\cdot) \text{ and } \partial_{x_1}u(1,\cdot) = e^{i\theta}\partial_{x_1}u(0,\cdot) \ \text{ in } \mathbb{R}^2\},$$

and

$$H^2_\theta(\mathcal{R}) = \{u \in H^2(\mathcal{R});\ u(1,\cdot) = e^{i\theta}u(0,\cdot) \text{ and } \partial_{x_1}u(1,\cdot) = e^{i\theta}\partial_{x_1}u(0,\cdot) \text{ in } \mathbb{R}^2\}.$$

We denote by $\tilde{\tau}$ the linear bounded operator from $H^2\left(0,T;H^2\left(\tilde{\Omega}\right)\right)$ into $L^2\left((0,T)\times(0,1), H^{3/2}(\partial\omega)\right)$, such that

$$\tilde{\tau}w = w_{|\tilde{\Sigma}} \text{ for } w \in C_0^\infty\left([0,T]\times(0,1), C^\infty(\overline{\omega})\right).$$

Then the space $\tilde{X}_\theta = \tilde{\tau}\left(H^2\left(0,T;H^2_\theta\left(\tilde{\Omega}\right)\right)\right)$ endowed with the norm

$$\| w \|_{\tilde{X}_\theta} = \inf\{\|W\|_{H^2\left(0,T;H^2(\tilde{\Omega})\right)};\ W \in H^2\left(0,T;H^2\left(\tilde{\Omega}\right)\right) \text{ such that } \tilde{\tau}W = w\},$$

is Hilbertian. The operator

$$\Lambda_{A,\theta} : h \in \tilde{X}_\theta \longmapsto (\partial_\nu + iA\cdot\nu)\, u \in L^2\left(\tilde{\Sigma}\right), \tag{1.9}$$

where u is the solution to (1.8), is bounded. The following result essentially claims that Theorem 1.1 remains valid upon substituting $\Lambda_{A_j,\theta}$ for A_j, $j = 1,2$, for θ arbitrary in $[0,2\pi)$.

Theorem 1.2 *Let A_1 and A_2 obey the conditions of Theorem 1.1 for $M > 0$. Then we may find a constant $C > 0$ depending only T, M and ω, such that the estimate*

$$\left\| \frac{\partial a_2}{\partial x_3} - \frac{\partial a_3}{\partial x_2} \right\|_{H^{-1}(\tilde{\Omega})} \leq C \|\Lambda_{A_2,\theta} - \Lambda_{A_1,\theta}\|^{\mu},$$

holds for every $\theta \in [0,2\pi)$.

It is clear that Theorem 1.1 yields uniqueness in the identification of the aligned magnetic field from the knowledge of the DN map.

2. Geometric optics solutions

In this section we define geometric optics solutions for the magnetic wave equation in $(0,T)\times\tilde{\Omega}$ with quasi-periodic boundary conditions. These functions are essential tools in the proof of Theorems 1.1 and 1.2. The main difficulty here is the quasi-periodic boundary conditions. We shall adapt the method suggested by Bellassoued and Ben joud in [2], for building geometric optics solutions to the magnetic wave equation in a bounded domain, to the framework of periodic media.

Let $v=v\left(t,x\right)\in\mathcal{C}^1\left([0,T]\,;L^2\left(\tilde{\Omega}\right)\right)\cap\mathcal{C}\left([0,T]\,;H^1\left(\tilde{\Omega}\right)\right)$ be a given solution of the following magnetic wave equation :

$$\begin{cases} \left(\partial_t^2-\Delta_A\right)v=0 & \text{in } \tilde{\Omega},\\ v(T,.)=0 & \text{in } \tilde{\Omega},\\ \partial_t v(T,.)=0 & \text{in } \tilde{\Omega},\\ v=h & \text{in } \tilde{\Sigma},\\ v(.,1,.)=e^{i\theta}v(.,0,.) & \text{in } (0,T)\times\omega,\\ \partial_{x_1}v(.,1,.)=e^{i\theta}\partial_{x_1}v(.,0,.) & \text{in } (0,T)\times\omega. \end{cases} \tag{2.1}$$

Let $u=u\left(t,x\right)\in\mathcal{C}^1\left([0,T]\,;L^2\left(\tilde{\Omega}\right)\right)\cap\mathcal{C}\left([0,T]\,;H^1\left(\tilde{\Omega}\right)\right)$ satisfy the conditions

$$u\left(0,.\right)=\partial_t u\left(0,.\right)=0 \text{ in } \tilde{\Omega},\quad u=0\ \text{ in } \tilde{\Sigma}, \tag{2.2}$$

and

$$u\left(.,1,.\right)-e^{i\theta}u\left(.,0,.\right)=\partial_{x_1}u\left(.,1,.\right)-e^{i\theta}\partial_{x_1}u\left(.,0,.\right)=0,\ \text{in } (0,T)\times\omega.$$

Then, from the Green formula, we have

$$\begin{aligned} \int_{\tilde{Q}}\left(\partial_t^2u-\Delta_A u\right)\overline{v}dxdt &= \int_{\tilde{Q}}u\overline{(\partial_t^2v-\Delta_A v)}dxdt-\int_{\tilde{\Sigma}}\left(\partial_\nu+iA\cdot\nu\right)u\overline{v}d\sigma_x dt\\ &= -\int_{\tilde{\Sigma}}\left(\partial_\nu+iA\cdot\nu\right)u\overline{v}d\sigma_x dt. \end{aligned} \tag{2.3}$$

2.1. Geometric optics solutions in periodic media

For all $r>0$, we take $\varrho>0$, such that

$$T>T-4\varrho>D_{\Omega'}\quad \text{ and }\quad \overline{\Omega'}\subset B(x_0,(T/2)-2\varrho),$$

and let $\phi_0\in\mathcal{C}_0^\infty\left(\mathbb{R}^2\right)$ such that

$$\operatorname{supp}\phi_0\subset\mathfrak{D}_\varrho, \tag{2.4}$$

where

$$\mathfrak{D}_\varrho=B\left(0,\frac{T}{2}\right)\setminus\overline{B}\left(0,\frac{T}{2}-2\varrho\right). \tag{2.5}$$

Now, for $\kappa'\in\mathbb{S}^1,\ \theta$ fixed in $[0,2\pi)$ and $\phi_\theta\in H_\theta^2\left((0,1)\times\mathbb{R}^2\right)$, we may introduce the following subspace of $H^2\left((0,1)\times\mathbb{R}^2\right)$,

$$\mathcal{H}_{\theta,\kappa'}^2\left(\mathfrak{D}_\varrho\right)=\{\phi=\phi_\theta\phi_0\in H^2\left((0,1)\times\mathbb{R}^2\right);\ \kappa'\cdot\nabla_{x'}\phi\in H^2\left((0,1)\times\mathbb{R}^2\right)\}$$

where

$$\phi_0 \in H^2\left(\mathbb{R}^2\right) \text{ satisfies (2.4)}$$

endowed with the natural norm :

$$\mathcal{N}_{\kappa'}\left(\phi\right) = \|\phi\|_{H^2((0,1)\times\mathbb{R}^2)} + \|\kappa' \cdot \nabla_{x'}\phi\|_{H^2((0,1)\times\mathbb{R}^2)}.$$

It is apparent from (2.4) and (2.5) that, for all $\phi \in \mathcal{H}^2_{\theta,\kappa'}\left(\mathfrak{D}_R\right)$

$$\operatorname{supp}\phi \cap \omega = \emptyset, \quad \left(\operatorname{supp}\phi \pm T\kappa'\right) \cap \omega = \emptyset, \quad \forall \kappa' \in \mathbb{S}^1. \tag{2.6}$$

Next, for all $t \in \mathbb{R}$ and $\phi \in \mathcal{H}^2_{\theta,\kappa'}\left(\mathfrak{D}_R\right)$, we put

$$\Phi\left(t,x\right) = \phi\left(x_1, x' + t\kappa'\right), \quad t \in \mathbb{R}, \left(x_1, x'\right) \in \mathbb{R}^3, \tag{2.7}$$

then it is clear that the function Φ is solution to the transport equation

$$\left(\partial_t - \kappa' \cdot \nabla_{x'}\right) \Phi\left(t,x\right) = 0, \quad \left(t,x\right) \in \mathbb{R} \times \mathbb{R}^3. \tag{2.8}$$

Having seen this, we define

$$b\left(t,x\right) = \exp\left(i\int_0^t \kappa' \cdot A'\left(x_1, x' + s\kappa'\right) ds\right). \tag{2.9}$$

where

$$A'\left(x\right) = \left\{ \begin{array}{ll} \left(a_2, a_3\right)\left(x\right), & \text{if } x \in \Omega, \\ 0, & \text{if not.} \end{array} \right.$$

It is easy to see that

$$\left(\partial_t - \kappa' \cdot \nabla_{x'} - i\kappa' \cdot A'\right) b\left(t,x\right) = 0, \quad \text{for all} \quad \left(t,x\right) \in \mathbb{R} \times \mathbb{R}^3. \tag{2.10}$$

Let us now prove the following lemma.

Lemma 2.1 *Assume that* $A \in W^{3,\infty}\left(\Omega; \mathbb{R}^3\right)$ *satisfies* (1.1). *Pick* Φ *(resp.* b*) as in* (2.7) *(resp.* (2.9)*). Then for all* $\theta \in [0, 2\pi)$ *and* $\lambda > 0$, *one can construct a solution* $u \in \mathcal{C}^1\left([0,T]; L^2\left(\tilde{\Omega}\right)\right) \cap \mathcal{C}\left([0,T]; H^1_\theta\left(\tilde{\Omega}\right)\right)$ *to the equation*

$$\left(\partial_t^2 - \Delta_A\right) u = 0, \quad \left(t,x\right) \in \tilde{Q}, \tag{2.11}$$

of the form

$$u\left(t,x\right) = \Phi\left(t,x\right) b\left(t,x\right) e^{i\lambda\left(x'\cdot\kappa' + t\right)} + \psi_\lambda\left(t,x\right),$$

where the correction term ψ_λ *satisfies* $\psi_\lambda\left(0,.\right) = \psi_\lambda\left(T,.\right) = 0$ *in* $\tilde{\Omega}$, $\psi_\lambda = 0$ *on* $\tilde{\Omega}$ *together with the quasi-periodic boundary conditions*

$$\psi_\lambda\left(.,1,.\right) - e^{i\theta}\psi_\lambda\left(.,0,.\right) = \partial_{x_1}\psi_\lambda\left(.,1,.\right) = e^{i\theta}\partial_{x_1}\psi_\lambda\left(.,0,.\right) = 0 \ \textit{ in } \ (0,T) \times \omega.$$

Moreover, the following estimate

$$\lambda \left\|\psi_\lambda\right\|_{L^2\left(\tilde{Q}\right)} + \left\|\nabla\psi_\lambda\right\|_{L^2\left(\tilde{Q}\right)} \leq C\mathcal{N}_{\kappa'}\left(\phi\right), \tag{2.12}$$

holds for some constant $C > 0$ *depending only on* $\tilde{\Omega}$, T *and* $\|A\|_{W^{3,\infty}(\Omega)}$.

Proof 2.1 *To prove our Lemma, it would be enough to show that if* ψ_λ *solves*

$$\begin{cases} \left(\partial_t^2 - \Delta_A\right)\psi_\lambda = \\ -\left(\partial_t^2 - \Delta_A\right)\left(\phi\left(x_1, x' + t\kappa'\right) b\left(t,x\right) e^{i\lambda\left(x'\cdot\kappa' + t\right)}\right) & \text{in } \tilde{Q}, \\ \psi_\lambda\left(0,.\right) = 0 & \text{in } \tilde{\Omega}, \\ \partial_t\psi_\lambda\left(0,.\right) = 0 & \text{in } \tilde{\Omega}, \\ \psi_\lambda = 0, & \text{on } \tilde{\Sigma}, \\ \psi_\lambda\left(.,1,.\right) = e^{i\theta}\psi_\lambda\left(.,0,.\right) & \text{in } (0,T)\times\omega, \\ \partial_{x_1}\psi_\lambda\left(.,1,.\right) = e^{i\theta}\partial_{x_1}\psi_\lambda\left(.,0,.\right) & \text{in } (0,T)\times\omega. \end{cases} \tag{2.13}$$

Then the estimate (2.12) *holds. We set*

$$k\left(t,x\right) = -\left(\partial_t^2 - \Delta_A\right)\left(\phi\left(x_1, x' + t\kappa'\right) b\left(t,x\right) e^{i\lambda\left(x'\cdot\kappa' + t\right)}\right), \quad (t,x) \in (0,T)\times\tilde{\Omega}. \tag{2.14}$$

Thus, we have

$$\begin{aligned} k\left(t,x\right) \;\; = \;\; & -e^{i\lambda\left(x'\cdot\kappa'+t\right)}\left(\partial_t^2 - \Delta_A\right)\left(\Phi\left(t,x\right) b\left(t,x\right)\right) \\ & -2i\lambda e^{i\lambda\left(x'\cdot\kappa'+t\right)}\left(\partial_t - \kappa'\cdot\nabla_{x'}\right)\left(\Phi\left(t,x\right)\right) \\ & -2i\lambda e^{i\lambda\left(x'\cdot\kappa'+t\right)}\left(\partial_t - \kappa'\cdot\nabla_{x'} - i\kappa'\cdot A'\right)\left(b\left(t,x\right)\right). \end{aligned}$$

Since $\Phi\left(t,x\right) = \phi\left(x_1, x' + t\kappa'\right)$ *and* $b\left(t,x\right)$ *are the respective solutions of* (2.8) *and* (2.10) *we deduce that*

$$k\left(t,x\right) = -e^{i\lambda\left(x'\cdot\kappa'+t\right)}\left(\partial_t^2 - \Delta_A\right)\left(\Phi\left(t,x\right) b\left(t,x\right)\right) = -e^{i\lambda\left(x'\cdot\kappa'+t\right)} k_0\left(t,x\right)$$

where k_0 *satisfies*

$$\left\|k_0\right\|_{L^2\left(\tilde{Q}\right)} + \left\|\partial_t k_0\right\|_{L^2\left(\tilde{Q}\right)} \leq C\mathcal{N}_{\kappa'}\left(\phi\right).$$

Since the coefficient A *does not depend on* t, *the function*

$$w\left(t,x\right) = \int_0^t \psi\left(s,x\right) ds,$$

solves the mixed hyperbolic problem (2.13) *with the right side*

$$k_1\left(t,x\right) = \int_0^t k\left(s,x\right) ds = \frac{1}{i\lambda}\int_0^t k_0\left(s,x\right)\partial_s\left(e^{i\lambda\left(x'\cdot\kappa'+s\right)}\right) ds.$$

Integrating by part with respect to s, *we conclude that*

$$\left\|k_1\right\|_{L^2\left(\tilde{Q}\right)} \leq \frac{C}{\lambda}\mathcal{N}_{\kappa'}\left(\phi\right),$$

and it follows from the energy estimate for w *that*

$$\left\|\psi_\lambda\right\|_{L^2\left(\tilde{Q}\right)} = \left\|\partial_t w\right\|_{L^2\left(\tilde{Q}\right)} \leq \frac{C}{\lambda}\mathcal{N}_{\kappa'}\left(\phi\right),$$

Since $\left\|k\right\|_{L^2\left(\tilde{Q}\right)} \leq C\mathcal{N}_{\kappa'}\left(\phi\right)$ *we obtain*

$$\left\|\nabla\psi_\lambda\right\|_{L^2\left(\tilde{Q}\right)} \leq C\mathcal{N}_{\kappa'}\left(\phi\right),$$

which completes the proof.

remark 1 *We have a similar result by replacing the condition* $\psi_\lambda\left(0,.\right) = 0$ *in* $\tilde{\Omega}$ *by* $\psi_\lambda\left(T,.\right) = 0$ *in* $\tilde{\Omega}$.

3. Stability estimate

3.1. Preliminary estimate

Let us introduce

$$A = A_2 - A_1,$$

where the function $A_j,\ j = 1, 2$, is as in Theorem (1.1). Recall that since $A_1 - A_2 = 0$ on $\partial\Omega$, we can extend $A = (a_1, a_2, a_3)$ to $H^1(\mathbb{R}^3)$ and we set

$$b(t,x) = \left(b_2\overline{b}_1\right)(t,x) = \exp\left(i\int_0^t \kappa' \cdot A'(x_1, x' + s\kappa')\,ds\right),$$

where $A' = (a_1, a_2)$.

The main purpose of this subsection is the following technical result.

Lemma 3.1 *Let $\theta \in [0, 2\pi)$ be fixed. For $j = 1, 2$, let $\phi_j \in \mathcal{H}^2_{\theta,\kappa'}(\mathfrak{D}_R)$. Then for any $\kappa' \in \mathbb{S}^1$, there exists a constant $C = C(M, \omega) > 0$ such that,*

$$\mathcal{I}_{\kappa'}(\phi_1, \phi_2) \leq C\left(\lambda\left\|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\right\| + \frac{1}{\lambda}\right)\mathcal{N}_{\kappa'}(\phi_1)\,\mathcal{N}_{\kappa'}(\phi_2). \tag{3.1}$$

where

$$\mathcal{I}_{\kappa'}(\phi_1, \phi_2) = \left|\int_0^T\int_0^1\int_{\mathbb{R}^2} \kappa' \cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1, x' + t\kappa')\,b(t,x)\,dxdt\right|.$$

Proof 3.1 *Lemma 2.1 guarantees the existence of geometric optics solution $u_2 \in \mathcal{C}^1\left([0,T]; L^2\left(\tilde{\Omega}\right)\right) \cap \mathcal{C}\left([0,T]; H^1_\theta\left(\tilde{\Omega}\right)\right)$ to the equation $\left(\partial_t^2 - \Delta_{A_2}\right)u$ in $\tilde{Q}$, with the form*

$$u_2(t,x) = \phi_2(x_1, x' + t\kappa')\,b_2(t,x)\,e^{i\lambda\left(x'\cdot\kappa' + t\right)} + \psi_{2,\lambda}(t,x), \tag{3.2}$$

where $\psi_{2,\sigma}$ satisfies

$$\begin{cases} \psi_{2,\lambda}(0,.) = 0 & \text{in } \tilde{\Omega}, \\ \partial_t\psi_{2,\lambda}(0,.) = 0 & \text{in } \tilde{\Omega}, \\ \psi_{2,\lambda} = 0 & \text{on } \tilde{\Sigma}, \\ \psi_{2,\lambda}(.,1,.) = e^{i\theta}\psi_{2,\lambda}(.,0,.) & \text{in } (0,T)\times\omega, \\ \partial_{x_1}\psi_{2,\lambda}(.,1,.) = e^{i\theta}\partial_{x_1}\psi_{2,\lambda}(.,0,.) & \text{in } (0,T)\times\omega, \end{cases} \tag{3.3}$$

and

$$\lambda\left\|\psi_{2,\lambda}\right\|_{L^2(\tilde{Q})} + \left\|\nabla\psi_{2,\lambda}\right\|_{L^2(\tilde{Q})} \leq C\mathcal{N}_{\kappa'}(\phi_2). \tag{3.4}$$

We denote by u_1, the solution of

$$\begin{cases} \left(\partial_t^2 - \Delta_{A_1}\right)u_1 = 0 & \text{in } \tilde{Q}, \\ u_1(0,.) = \partial_t u_1(0,.) = 0 & \text{in } \tilde{\Omega}, \\ u_1 = u_2 = f_{\lambda,2} & \text{on } \tilde{\Sigma}, \\ u_1(.,1,.) = e^{i\theta}u_1(.,0,.) & \text{in } \check{\Omega}, \\ \partial_{x_1}u_1(.,1,.) = e^{i\theta}\partial_{x_1}u_1(.,0,.) & \text{in } \tilde{\Omega}. \end{cases} \tag{3.5}$$

Further, as $u_1 - u_2 = 0$ *on* $\tilde{\Sigma}$, *we get from* (3.5) *and* (3.2) *that* $u = u_1 - u_2 \in \mathcal{C}^1\left([0,T]; L^2\left(\tilde{\Omega}\right)\right) \cap \mathcal{C}\left([0,T]; H^1_\theta\left(\tilde{\Omega}\right)\right)$ *verifies*

$$\left\{\begin{array}{ll} \left(\partial_t^2 - \Delta_{A_1}\right) u = 2iA\cdot\nabla u_2 + V u_2 & \text{in } \tilde{Q},\\ u(0,.) = \partial_t u(0,.) = 0 & \text{in } \tilde{\Omega},\\ u = 0 & \text{on } \tilde{\Sigma},\\ u(.,1,.) = e^{i\theta} u(.,0,.) & \text{in } \tilde{\Omega},\\ \partial_{x_1} u(.,1,.) = e^{i\theta}\partial_{x_1} u(.,0,.) & \text{in } \tilde{\Omega}, \end{array}\right.$$

where we have set

$$V = i\,div(A) - (A_2\cdot A_2 - A_1\cdot A_1).$$

Next let $v \in \mathcal{C}^1\left([0,T]; L^2\left(\tilde{\Omega}\right)\right) \cap \mathcal{C}\left([0,T]; H^1_\theta\left(\tilde{\Omega}\right)\right)$ *be a solution of the wave equation* $\left(\partial_t^2 - \Delta_{A_1}\right) v = 0,$ *in* $\tilde{Q}$, *having the form*

$$v(t,x) = \phi_1(x_1, x' + t\kappa')\, b_1(t,x)\, e^{i\lambda\left(x'\cdot\kappa' + t\right)} + \psi_{1,\lambda}(t,x), \tag{3.6}$$

where $\psi_{1,\lambda}$ *satisfies*

$$\left\{\begin{array}{ll} \psi_{1,\lambda}(T,.) = 0 & \text{in } \tilde{\Omega},\\ \psi_{1,\lambda} = 0 & \text{on } \tilde{\Sigma},\\ \psi_{1,\lambda}(.,1,.) = e^{i\theta}\psi_{1,\lambda}(.,0,.) & \text{in } (0,T)\times\omega,\\ \partial_{x_1}\psi_{1,\lambda}(.,1,.) = e^{i\theta}\partial_{x_1}\psi_{1,\lambda}(.,0,.) & \text{in } (0,T)\times\omega, \end{array}\right. \tag{3.7}$$

and

$$\lambda\left\|\psi_{1,\lambda}\right\|_{L^2\left(\tilde{Q}\right)} + \left\|\nabla\psi_{1,\lambda}\right\|_{L^2\left(\tilde{Q}\right)} \leq C\mathcal{N}_{\kappa'}(\phi_1). \tag{3.8}$$

Set

$$g_\lambda(t,x) = \phi_1(x_1, x' + t\kappa')\, b_1(t,x)\, e^{i\lambda\left(x'\cdot\kappa' + t\right)}, \quad (t,x) \in \tilde{\Sigma},$$

In light of (2.3) , *we deduce from* (3.5), *the following orthogonality identity*

$$\begin{aligned} \int_{\tilde{Q}} 2iA\cdot\nabla u_2\overline{v}\,dxdt + \int_{\tilde{Q}} V(x)\, u_2\overline{v}\,dxdt &= -\int_{\tilde{\Sigma}} \left(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\right)(f_\lambda)\,\overline{g_\lambda}\,d\sigma dt\\ &= -\langle\left(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\right)(f_\lambda), \overline{g_\lambda}\rangle. \end{aligned} \tag{3.9}$$

On the other hand by (3.2) *and* (3.6) *we have*

$$\begin{aligned} \int_{\tilde{Q}} 2iA\cdot\nabla u_2\overline{u_1}\,dxdt &= -\int_{\tilde{Q}} 2\lambda\kappa'\cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1, x' + t\kappa')\left(b_2\overline{b}_1\right)(t,x)\,dxdt\\ &\quad + \mathcal{I}_\lambda, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \mathcal{I}_\lambda &= \int_{\tilde{Q}} 2iA\cdot\nabla\left(\Phi_2(t,x)\, b_2(t,x)\right)\overline{\Phi}_1(t,x)\,\overline{b}_1(t,x)\,dxdt\\ &\quad + \int_{\tilde{Q}} 2iA\cdot\nabla\left(\Phi_2(t,x)\, b_2(t,x)\right) e^{i\lambda\left(x'\cdot\kappa' + t\right)}\overline{\psi}_{1,\lambda}(t,x)\,dxdt\\ &\quad + \int_{\tilde{Q}} 2iA\cdot\nabla\psi_{2,\lambda}(t,x)\,\overline{\Phi}_1(t,x)\,\overline{b}_1(t,x)\, e^{i\lambda\left(x'\cdot\kappa' + t\right)}\\ &\quad + \int_{\tilde{Q}} 2iA\cdot\nabla\psi_{2,\lambda}(t,x)\,\overline{\psi}_{1,\lambda}(t,x)\,dxdt\\ &\quad - \int_{\tilde{Q}} 2\lambda\kappa'\cdot A'(x)\, b_2(t,x)\,\Phi_2(t,x)\,\overline{\psi}_{1,\lambda} e^{i\lambda\left(x'\cdot\kappa' + t\right)}\,dxdt. \end{aligned}$$

Using (3.4) *and* (3.8)*, we obtain that*

$$|\mathcal{I}_\lambda| \leq C\mathcal{N}_{\kappa'}(\phi_2)\mathcal{N}_{\kappa'}(\phi_1). \tag{3.11}$$

From this and (3.9)-(3.10)*, it follows that*

$$\begin{aligned}&\lambda\left|\int_{\tilde{Q}} \kappa' \cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1, x' + t\kappa')\left(b_2\overline{b}_1\right)(t,x)\,dxdt\right| \\ &\leq C(\left|\int_{\tilde{Q}} Vu_2\overline{v}dxdt\right| + \left|\int_{\Sigma_1}(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta})(f_\lambda)\overline{g_\lambda}d\sigma dt\right| \\ &\qquad + \mathcal{N}_{\kappa'}(\phi_2)\mathcal{N}_{\kappa'}(\phi_1)).\end{aligned} \tag{3.12}$$

Moreover, by (3.2)*,* (3.4)*,* (3.6) *and* (3.8)*, one gets*

$$\left|\int_{\tilde{Q}} V(x)\,u_2\overline{v}dxdt\right| \leq C\mathcal{N}_{\kappa'}(\phi_2)\mathcal{N}_{\kappa'}(\phi_1). \tag{3.13}$$

By a trace inequality, we have

$$\begin{aligned}\left|\int_{\tilde{\Sigma}}(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta})(f_\lambda)\overline{g_\lambda}d\sigma dt\right| &= \left|\langle(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta})(f_\lambda), g_\lambda\rangle_{L^2(\tilde{\Sigma})}\right| \\ &\leq \|(\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta})(f_\lambda)\|_{L^2(\tilde{\Sigma})}\|g_\lambda\|_{L^2(\tilde{\Sigma})} \\ &\leq \|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\|\,\|f_\lambda\|_{\tilde{X}_\theta}\|g_\lambda\|_{L^2(\tilde{\Sigma})} \\ &\leq C\lambda^2\|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\| \\ &\qquad \mathcal{N}_{\kappa'}(\phi_1)\mathcal{N}_{\kappa'}(\phi_2).\end{aligned} \tag{3.14}$$

From (3.12)-(3.13) *and* (3.14)*, we derive for* λ *sufficiently large*

$$\begin{aligned}&\left|\int_0^T\int_0^1\int_{\mathbb{R}^2}\kappa' \cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1, x' + t\kappa')\,b(t,x)\,dxdt\right| \\ &\leq C\left(\lambda\|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\| + \frac{1}{\lambda}\right)\mathcal{N}_{\kappa'}(\phi_1)\mathcal{N}_{\kappa'}(\phi_2).\end{aligned} \tag{3.15}$$

This completes the proof of the lemma.

3.2. X-ray transform estimate

The X-ray transform is an integral transform, defined by integrating over lines. More precisely, if $\kappa' \in \mathbb{S}^1$ and f is a function defined on $\mathbb{R}^3$, then the X-ray transform of f in the direction κ' is the function $\mathcal{P}f$ defined by

$$(\mathcal{P}f)(\kappa', x) = \int_{\mathbb{R}} f(x_1, x' + s\kappa')\,ds,\ \ x = (x_1, x') \in \mathbb{R}^3. \tag{3.16}$$

It is easy to see that $(\mathcal{P}f)(\omega', x)$ does not change if x' is moved in the direction ω'. Therefore we normally restrict x' to $\tilde{\imath}^{\perp} = \{\varsigma \in \mathbb{R}^2;\ \varsigma \cdot \omega' = 0\}$. For $j = 1, 2, 3$, let us introduce the following notations

$$\rho_j(x) = \kappa' \cdot \frac{\partial A'}{\partial x_j}(x) = \sum_{i=2}^{3}\kappa_i\frac{\partial a_i}{\partial x_j}(x),\ \ x \in \mathbb{R}^3. \tag{3.17}$$

and

$$\mathfrak{D}_\varrho^+(\kappa') = \{x' \in \mathfrak{D}_\varrho,\ x' \cdot \kappa' > 0\}.$$

Then, the X-ray transform stability estimate of the functions ρ_j is as follows

Lemma 3.2 *Let $M > 0$, and let A_j, for $j = 1, 2$, be as in theorem 1.1. Then, there exists constants $C > 0$ and $\lambda_0 > 0$ such that for all $\kappa' \in \mathbb{S}^1$ and all $\phi = \phi_\theta\phi_0 \in \mathcal{H}^2_{\kappa',\theta}(\mathfrak{D}_\varrho)$ satisfying supp $(\phi_0) \subset \mathfrak{D}^+_\varrho$ and $\partial_j\phi \in \mathcal{H}^2_{\kappa',\theta}(\mathfrak{D}_\varrho)$, the estimate*

$$\left|\int_0^1 e^{-i2k\pi x_1}\int_{\mathbb{R}^2}\phi^2(x)(\mathcal{P}\rho_j)(\kappa',x)\exp\left(i\int_{\mathbb{R}}\kappa'\cdot A'(x_1,x'+s\kappa')\,ds\right)dx\right| \leq$$
$$C\left(\lambda\left\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\right\|+\frac{1}{\lambda}\right)\mathcal{N}_{\kappa'}(\phi)\,\mathcal{N}_{\kappa'}(\partial_j\phi),$$

holds for any $\lambda \geq \lambda_0$, $k \in \mathbb{Z}$ and $j = 2, 3$.

Proof 3.2 *For $\phi_1, \phi_2 \in \mathcal{H}^2_{\kappa',\theta}(\mathfrak{D}_\varrho)$, we have*

$$\begin{aligned}
&\int_0^T\int_0^1\int_{\mathbb{R}^2}\kappa'\cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1,x'+t\kappa')\,b(t,x)\,dx'dx_1dt\\
=\;&\int_0^T\int_0^1\int_{\mathbb{R}^2}\kappa'\cdot A'(x_1,x'-t\kappa')\left(\phi_2\overline{\phi}_1\right)(x)\,b(t,x_1,x'-t\kappa')\,dx'dx_1dt\\
=\;&\int_0^1\int_{\mathbb{R}^2}\left(\phi_2\overline{\phi}_1\right)(x)\int_0^T\kappa'\cdot A'(x_1,x'-t\kappa')\\
=\;&\exp\left(i\int_0^t\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)dtdx'dx_1\\
=\;&i\int_0^1\int_{\mathbb{R}^2}\left(\phi_2\overline{\phi}_1\right)(x)\int_0^T\frac{d}{dt}\exp\left(i\int_0^t\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)dtdx_1dx'\\
=\;&i\int_0^1\int_{\mathbb{R}^2}\left(\phi_2\overline{\phi}_1\right)(x)\left[\exp\left(i\int_0^T\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)-1\right]dx. \qquad (3.18)
\end{aligned}$$

We choose ϕ_1 and ϕ_2 such that $\phi_2(x) = e^{-i2k\pi x_1}\phi(x)$, $\phi_1 = \partial_j\overline{\phi}$, $j \in \{2, 3\}$ and integrating by parts, so (3.18) *yields*

$$\begin{aligned}
&\int_0^T\int_0^1\int_{\mathbb{R}^2}\kappa'\cdot A'(x)\left(\phi_2\overline{\phi}_1\right)(x_1,x'+t\kappa')\,b(t,x)\,dxdt\\
=\;&\frac{i}{2}\int_0^1\int_{\mathbb{R}^2}e^{-i2k\pi x_1}\phi^2(x)\frac{\partial}{\partial x_j}\left[\exp\left(i\int_0^T\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)\right]dx\\
=\;&-\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}e^{-i2k\pi x_1}\phi^2(x)\frac{\partial}{\partial x_j}\left(\int_0^T\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)\\
&\qquad\exp\left(i\int_0^T\kappa'\cdot A'(x_1,x'-s\kappa')\,ds\right)dx. \qquad (3.19)
\end{aligned}$$

Since the support of A' is contained in $\mathbb{R}\times B(0,T/2-2\varrho)$, we have

$$\int_0^T\kappa'\cdot A'(x_1,x'-s\kappa')\,ds = \int_{\mathbb{R}}\kappa'\cdot A'(x_1,x'-s\kappa')\,ds, \qquad (3.20)$$

for all $x' \in \mathfrak{D}^+_\varrho(\kappa')$. In fact, for all $s \geq T$ and $x' \in \mathfrak{D}_R$ it is easy to see that $(x_1, x'+s\kappa') \notin$ supp (A'), for each $x_1 \in [0,1]$. Therefore we have

$$\int_0^T\kappa'\cdot A'(x_1,x'-s\omega')\,ds = \int_0^\infty\kappa'\cdot A'(x_1,x'-s\kappa')\,ds,\;\;(x_1,x') \in [0,1]\times\mathfrak{D}^+_\varrho(\kappa'). \qquad (3.21)$$

On the other hand, if $s \leq 0$ and $x' \in \mathfrak{D}_\varrho^+$ it holds true that $|x' - s\kappa'|^2 = |x'|^2 + s^2 - 2sx' \cdot \kappa' \geq (T/2 - 2\varrho)^2$ hence $A(x_1, x' + s\kappa') = 0$. This and (3.21) entail (3.20). Further, upon inserting (3.20) into the equation (3.19), we obtain

$$\begin{aligned}
&\int_0^T \int_0^1 \int_{\mathbb{R}^2} \kappa' \cdot A'(x) \left(\phi_2 \overline{\phi}_1\right)(x_1, x' + t\kappa')\, b(t,x)\, dxdt \\
&= -\frac{1}{2} \int_0^1 \int_{\mathbb{R}^2} e^{-i2k\pi x_1} \phi^2(x) \frac{\partial}{\partial x_j} \left(\int_{\mathbb{R}} \kappa' \cdot A'(x_1, x' - s\omega')\, ds \right) \\
&\qquad\qquad\qquad\qquad \exp\left(i \int_{\mathbb{R}} \kappa' \cdot A'(x_1, x' - s\kappa')\, ds \right) dx \\
&= -\frac{1}{2} \int_0^1 \int_{\mathbb{R}^2} e^{-i2k\pi x_1} \phi^2(x)\, \mathcal{P}(\rho_j)(\kappa', x) \exp\left(i \int_{\mathbb{R}} \kappa' \cdot A'(x_1, x' - s\kappa')\, ds \right) dx,
\end{aligned}$$

where ρ_j is given by (3.17). From this and Lemma 3.1, we obtain for any $\lambda > \lambda_0$ that

$$\left| \int_0^1 e^{-i2k\pi x_1} \int_{\mathbb{R}^2} \phi^2(x)\, \mathcal{P}(\rho_j)(\kappa', x) \exp\left(i \int_{\mathbb{R}} \kappa' \cdot A'(x_1, x' - s\kappa')\, ds \right) dx' dx_1 \right| \leq$$

$$C \left(\lambda \left\| \Lambda_{A_1,\theta} - \Lambda_{A_2,\theta} \right\| + \frac{1}{\lambda} \right) \mathcal{N}_{\kappa'}(\phi)\, \mathcal{N}_{\kappa'}(\partial_j \phi).$$

The proof is then complete.

3.3. Aligned magnetic field estimation

In this subsection, we estimate the the partial Fourier transform of the aligned magnetic field, in terms of the DN map. To this end, we denoted by $\widehat{f}$ the partial Fourier transform of the function f with respect to the variable $x' \in \omega$, i.e

$$\widehat{f}(x_1, \xi') = (2\pi)^{-1} \int_{\mathbb{R}^2} f(x_1, x')\, e^{-ix' \cdot \xi'} dx', \quad \xi' \in \mathbb{R}^2,\ x_1 \in \mathbb{R}.$$

Further, setting $\omega'^{\perp} = \{\varsigma \in \mathbb{R}^2;\ \varsigma \cdot \omega' = 0\}$, we recall from the definition (3.16) that

Lemma 3.3 *Let $f \in L^1\left(\mathbb{R}^3\right)$ and $\kappa' \in \mathbb{S}^1$. Then $(\mathcal{P}f)(\kappa', .) \in L^1\left(\mathbb{R} \times {\kappa'}^{\perp}\right)$ and*

$$\left(\widehat{(\mathcal{P}f)(\kappa', .)}\right)(x_1, \xi') = (2\pi)^{-1} \int_{{\kappa'}^{\perp}} e^{-ix' \cdot \xi'} (\mathcal{P}f)(\kappa', x_1, x')\, dx' = \widehat{f}(x_1, \xi'),$$

for all $\xi' \in {\kappa'}^{\perp}$.

Let us now estimate the Fourier transform of the aligned magnetic field σ_{23}, where

$$\sigma_{ij} = \frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i}, \quad i, j = 1, 2, 3.$$

More precisely, with the aid of Lemma 3.2 and lemma 3.3 we may establish the following result.

Lemma 3.4 *Let $M > 0$, and let A_j, for $j = 1,2$, be as in theorem (1.1). Then, there exists a constant $C > 0$ such that for any $\lambda > \lambda_0$ the following estimate*

$$\left| \int_0^1 e^{i2k\pi x_1} \widehat{\sigma}_{23}\left(x_1, \xi'\right) dx_1 \right| \leq C\left(\lambda \left\|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\right\| + \frac{1}{\lambda}\right) \langle (k,\xi') \rangle^5, \qquad (3.22)$$

holds uniformly in $k \in \mathbb{Z}$ and $\xi' \in \mathbb{R}^2$, with $\langle (k,\xi') \rangle = \left(1 + k^2 + |\xi'|^2\right)^{1/2}$ and where $\widehat{\sigma}_{23}$ denotes the partial Fourier transform of σ_{23} with respect to x'.

Proof 3.3 *We fix $z_0' \in \kappa'^{\perp} \cap B\left(0, T/2 - \varrho\right)$. Let $h \in \mathcal{C}_0^\infty(\mathbb{R})$ be supported in $(0, \varrho/4)$ and satisfy the condition*

$$\int_{\mathbb{R}} h^2(t)\, dt = 1.$$

Let

$$\eta_{z_0'} = \left(\frac{T}{2} - \frac{\varrho}{2}\right)^2 - |z_0'|^2, \quad z_1' = z_0' + \eta_{z_0'} \kappa'.$$

It is not difficult to check that

$$B\left(z_1', \varrho/2\right) \subset \mathfrak{D}_\varrho^+\left(\kappa'\right).$$

Let $\beta_0 \in \mathcal{C}_0^\infty\left(\kappa'^{\perp} \cap B\left(z_0', \varrho/4\right)\right)$ be nonnegative and for $y = (y_1, y') \in \mathbb{R}^3$, put

$$\phi_\theta(y) = e^{i\theta y_1} \exp\left(-\frac{i}{2} \int_{\mathbb{R}} \kappa' \cdot A'\left(y_1, y' - s\kappa'\right) ds\right), \ y = (y_1, y') \in \mathbb{R}^3,$$

and

$$\phi_0(y') = h\left(y' \cdot \kappa' - \eta_{z_0'}\right) e^{-\frac{i}{2} y' \cdot \xi'} \beta_0^{1/2}\left(y' - \left(y' \cdot \kappa'\right)\kappa'\right), \ y' \in \mathbb{R}^2.$$

It is apparent that

$$\mathit{supp}\left(\phi_0\right) \subset B\left(z_1', \varrho/2\right) \subset \mathfrak{D}_\varrho^+\left(\kappa'\right).$$

Set

$$\phi(y) = \phi_\theta(y)\, \phi_0(y'), \ y = (y_1, y') \in \mathbb{R}^3. \qquad (3.23)$$

It is clear that $\phi \in \mathcal{H}^2_{\kappa',\theta}\left(\mathfrak{D}_\varrho\right)$. By performing the change of variable $y' = x' + t\kappa' \in \kappa'^{\perp} \oplus \mathbb{R}\kappa'$ in the following integral, we get upon recalling that $\xi' \in \kappa'^{\perp}$, that

$$\begin{aligned}
& \int_0^1 \int_{\mathbb{R}^2} e^{-i2k\pi x_1} \phi^2(x) \left(\mathcal{P}\rho_j\right)\left(\kappa', x\right) \exp\left(i \int_{\mathbb{R}} \kappa' \cdot A'\left(x_1, x' - s\kappa'\right) ds\right) dx \\
= & \int_0^1 \int_{\mathbb{R}} \int_{\kappa'^{\perp}} e^{-i2k\pi x_1} \phi^2\left(x_1, x' + t\kappa'\right) \left(\mathcal{P}\rho_j\right)\left(\kappa', x_1, x' + t\kappa'\right) \\
& \qquad \exp\left(i \int_{\mathbb{R}} \omega' \cdot A'\left(x_1, x' - s\kappa'\right) ds\right) dx'dtdx_1 \\
= & \int_0^1 e^{-i2k\pi x_1} \int_{\mathbb{R}} \int_{\kappa'^{\perp}} h^2\left(t - \eta_{z_0'}\right) e^{-ix'\cdot\xi'} \beta_0(x') \left(\mathcal{P}\rho_j\right)\left(\kappa', x_1, x'\right) dx'dtdx_1 \\
= & \int_0^1 e^{-i2k\pi x_1} \int_{\kappa'^{\perp}} e^{-ix'\cdot\xi'} \beta_0(x') \left(\mathcal{P}\rho_j\right)\left(\kappa', x_1, x'\right) dx'dx_1.
\end{aligned}$$

It follows from this and Lemma 3.2 that

$$\begin{aligned}
& \left| \int_0^1 e^{-i2k\pi x_1} \int_{\kappa'^{\perp}} e^{-ix'\cdot\xi'} \beta_0(x') \left(\mathcal{P}\rho_j\right)\left(\kappa', x_1, x'\right) dx'dx_1 \right| \\
& \qquad \leq C\left(\lambda \left\|\Lambda_{A_1,\theta} - \Lambda_{A_2,\theta}\right\| + \frac{1}{\lambda}\right) \mathcal{N}_{\kappa'}(\phi)\, \mathcal{N}_{\kappa'}\left(\partial_j \phi\right).
\end{aligned}$$

As ϕ is given by (3.23) *and* $\mathcal{N}_{\kappa'}(\phi) = \|\phi\|_{H^2((0,1)\times\mathbb{R}^2)} + \|\kappa'\cdot\nabla_{x'}\phi\|_{H^2((0,1)\times\mathbb{R}^2)}$, *an elementary calculation gives for any* $\xi' \in \kappa'^{\perp}$

$$\mathcal{N}_{\kappa'}(\phi)\,\mathcal{N}_{\kappa'}(\partial_j\phi) \le C\langle (k,\xi')\rangle^5,$$

where $C>0$ *is independent of* k *and* ξ'.
From the last two inequalities we derive for all $\xi' \in \omega'^{\perp}$ *and* $k\in\mathbb{Z}$ *that*

$$\begin{aligned}\left|\int_0^1 e^{-i2k\pi x_1}\int_{\kappa'^{\perp}} e^{-ix'\cdot\xi'}\left(\mathcal{P}\rho_j\right)(\kappa',x)\,dx'dx_1\right| \\ \le C\left(\lambda\left\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\right\|+\tfrac{1}{\lambda}\right)\langle(k,\xi')\rangle^5.\end{aligned} \tag{3.24}$$

By Lemma 3.3 we have

$$\left|\int_0^1 e^{-i2k\pi x_1}\widehat{\rho_j}(x_1,\xi')\,dx_1\right| \le C\left(\lambda\left\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\right\|+\frac{1}{\lambda}\right)\langle(k,\xi')\rangle^5.$$

In view of the identity $\xi'\cdot\omega'=0$ *we have for* $j=2,3$,

$$\widehat{\rho_j}(.,\xi') = \sum_{i=2}^{3}\kappa_i\xi_j\widehat{a}_i(.,\xi') = \sum_{i=2}^{3}\kappa_i\left(\xi_j\widehat{a}_i(.,\xi')-\xi_i\widehat{a}_j(.,\xi')\right) = \sum_{i=2}^{3}\kappa_i\widehat{\sigma}_{ij}(.,\xi').$$

Since $\kappa'\in\mathbb{S}^1$ *is arbitrary, we get, for any* $\xi'\in\mathbb{R}^2$ *and* $k\in\mathbb{Z}$ *that*

$$\left|\int_0^1 e^{-i2k\pi x_1}\widehat{\sigma_{23}}(x_1,\xi')\,dx_1\right| \le C\left(\lambda\left\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\right\|+\frac{1}{\lambda}\right)\langle(k,\xi')\rangle^5,$$

proving the result.

Having established Lemma 3.4 we may now terminate the proof of Theorem 1.2.

3.4. Proof of the Theorem 1.2

For simplicity, we use the following notation

$$\widehat{\mathfrak{b}}(\xi',k) = \langle\widehat{\sigma_{23}}(\xi'),\phi_k\rangle_{L^2(0,1)} = \int_0^1 e^{-i2k\pi x_1}\widehat{\sigma_{23}}(x_1,\xi')\,dx_1,$$

where $\phi_k(x_1) = e^{-i2k\pi x_1}$ for all $k\in\mathbb{Z}$. Then, by the Parseval-Plancherel theorem, we find that

$$\begin{aligned}\|\sigma_{23}\|^2_{H^{-1}(\tilde{\Omega})} &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}\langle(\xi',k)\rangle^{-2}\left|\widehat{\mathfrak{b}}(\xi',k)\right|^2 d\xi' \\ &= \int_{\mathbb{R}^3}\langle(\xi',k)\rangle^{-2}\left|\widehat{\mathfrak{b}}(\xi',k)\right|^2 d\xi'd\mu(k),\end{aligned} \tag{3.25}$$

where $\mu = \sum_{n\in\mathbb{Z}}\delta_n$. Using (3.22) we get

$$\begin{aligned}\|\sigma_{23}\|^2_{H^{-1}((0,1)\times\mathbb{R}^2)} &= \int_{|(\xi',k)|\le R}\langle(\xi',k)\rangle^{-2}\left|\widehat{\mathfrak{b}}(\xi',k)\right|^2 d\xi'd\mu(k) \\ &\quad + \int_{|(\xi',k)|>R}\langle(\xi',k)\rangle^{-2}\left|\widehat{\mathfrak{b}}(\xi',k)\right|^2 d\xi'd\mu(k) \\ &\le C\left(\left(\lambda^2\left\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\right\|^2+\frac{1}{\lambda^2}\right)R^{13}+\frac{1}{R^2}\right).\end{aligned}$$

Choosing

$$\lambda^2 = R^{15}, \tag{3.26}$$

we obtain, that

$$\|\sigma_{23}\|_{H^{-1}((0,1)\times\mathbb{R}^2)} \leq C\left(R^{\ell}\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\|+\frac{1}{R}\right). \tag{3.27}$$

for some positive constant ℓ. The arguments above are valid for $\lambda \geq \lambda_0$. In light of (3.26) we need to take λ sufficiently large. So there exists a $\gamma \geq 0$ such that if $\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\| \leq \gamma$ and $R = \|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\|^{-1/(\ell+1)}$, we have $\lambda > \lambda_0$ and by (3.27) we obtain

$$\|\sigma_{23}\|_{H^{-1}((0,1)\times\mathbb{R}^2)} \leq C\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\|^{\mu},$$

where $\mu = 1/(\ell+1)$. Now if $\|\Lambda_{A_1}-\Lambda_{A_2}\| \geq \gamma$, it holds true that

$$\|\sigma_{23}\|_{H^{-1}((0,1)\times\mathbb{R}^2)} \leq \frac{2M}{\gamma^{1/(k+1)}}\gamma^{1/(k+1)} \leq \frac{2M}{\gamma^{\mu}}\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\|^{\mu}. \tag{3.28}$$

Thus it follows from (3.27) and (3.28) that,

$$\|\sigma_{23}\|_{H^{-1}((0,1)\times\mathbb{R}^2)} \leq C\|\Lambda_{A_1,\theta}-\Lambda_{A_2,\theta}\|^{\mu}.$$

This ends the proof of Theorem 1.2.

4. The Floquet decomposition

The idea here is to pass from the boundary operator Λ_A, thanks to the partial Floquet-Bloch-Gelfand (abbreviated to FBG in the following) Transform, to the family of operators $\{\Lambda_{A,\theta},\ \theta \in (0,2\pi)\}$. To that end, we start by recalling the definition of the partial FBG transform.

Let $f \in C_0^{\infty}(\mathbb{R}\times Y)$, where Y denotes a C^2 open subset or submanifold of $\mathbb{R}^n$, $n \in \mathbb{N}^*$. We define the partial FBG transform $\mathcal{U}$ by

$$\tilde{f}_\theta(x_1,y) = (\mathcal{U}f)_\theta(t,x) = \sum_{k=-\infty}^{+\infty} e^{-ik\theta} f(x_1+k,y),\ (x_1,y)\in\mathbb{R}\times Y,\ \theta\in[0,2\pi). \tag{4.1}$$

Then

$$\tilde{f}_\theta(x_1+1,y) = e^{i\theta}\tilde{f}_\theta(x_1,y),\ (x_1,y)\in\mathbb{R}\times Y,\ \theta\in[0,2\pi), \tag{4.2}$$

and

$$\left(\mathcal{U}\frac{\partial^m f}{\partial z^m}\right)_\theta = \frac{\partial^m \tilde{f}_\theta}{\partial z^m},\ m\in\mathbb{N}^*,\ \theta\in[0,2\pi), \tag{4.3}$$

We turn now to studying the problem (1.2). We will show that we can decompose the Cauchy problem (1.2) into a IBVP with quasi-periodic boundary conditions of the form (1.8).

Proposition 4.1 *Let $A \in W^{3,\infty}(0,T;W^{2,\infty}(\Omega))$ fulfill* (1.1) *and let $g \in X_0$. Then u is solution to* (1.2) *if and only if each $\tilde{u}_\theta = (\mathcal{U}u)_\theta \in \check{Z}_\theta = L^2\left(0,T;H^1_\theta\left(\tilde{\Omega}\right)\right) \cap H^1\left(0,T;L^2\left(\tilde{\Omega}\right)\right)$, $\theta \in [0,2\pi)$, is solution to the following IBVP*

$$\begin{cases} (\partial_t^2 - \Delta_A)v = 0 & \text{in } \tilde{Q}, \\ v(0,\cdot) = 0 & \text{in } \tilde{\Omega}, \\ \partial_t v(0,\cdot) = 0 & \text{in } \tilde{\Omega}, \\ v = \tilde{g}_\theta & \text{on } \tilde{\Sigma}. \end{cases} \tag{4.4}$$

Armed with Proposition 4.1, we may now decompose Λ_A in terms of the fibered boundary operators $\Lambda_{A,\theta}$, $\theta \in [0,2\pi)$.

Proposition 4.2 *Let A be the same as in Proposition 4.1. Then we have :*

$$\mathcal{U}\Lambda_A\mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} \Lambda_{A,\theta}\frac{d\theta}{2\pi}.$$

Proof 4.1 *For every $\theta \in [0,2\pi)$, we know that the IBVP* (4.4) *admits a unique solution $\mathfrak{s}_\theta(\tilde{g}_\theta)$. Further, it holds true that $\Lambda_{V,\theta} = \tilde{\tau}_1 \circ \mathfrak{s}_\theta$, where we recall that $\tilde{\tau}_1$ is the linear bounded operator from $L^2((0,T)\times(0,1),H^2(\tilde{\Omega}))\cap H^1(0,T;L^2(\tilde{\Omega}))$ into $\check{X}_1 = L^2(\tilde{\Sigma})\times L^2(\tilde{\Omega})$, obeying*

$$\tilde{\tau}_1 w = (\partial_\nu + iA\cdot\nu)\, w_{|\tilde{\Sigma}} \text{ for } w \in C_0^\infty((0,T)\times(0,1),C^\infty(\overline{\omega})).$$

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