Rubrique

Inverse heat source problem for a coupled hyperbolic-parabolic system

Bellassoued Mourad and Riahi Bochra

LASMIN, University of Tunis El Manar, National engineering school of Tunis, ENIT, Tunisia, mourad.bellassoued@fsb.rnu.tn
Department of Mathematics, University of Carthage, Bizerte, Tunisia
riahi.bochra@gmail.com

RÉSUMÉ. Dans ce papier, on a prouvé une estimation de stabilité de type Höldérienne pour un problème inverse de détermination du terme source de l'équation de la chaleur à l'aide d'une inégalité de Carleman pour un système d'équations hyperbolique-parabolique couplé.

ABSTRACT. In this paper we consider a coupled system of mixed hyperbolic-parabolic type which describes the Biot consolidation model in poro-elasticity. Using a local Carleman estimate for a coupled hyperbolic-parabolic system, we prove the uniqueness and a Hölder stability in determining the heat source by a single measurement of solution over $\omega \times (0,T)$, where $T > 0$ is a sufficiently large time and a suitable subbdomain $\omega \subset \Omega$ such that $\partial\omega \supset \partial\Omega$.

MOTS-CLÉS : Problème inverse, estimation de Carleman, système couplet

KEYWORDS : Inverse problem, Carleman estimate, coupled system

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ an open and bounded domain with $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$, and let t and $x = (x_1, x_2, x_3) \in \Omega$ denote the time variable and the spatial variable respectively. Given $T > 0$, we consider a coupled hyperbolic-parabolic system

$$\begin{cases} \mathbf{u}_{tt} - \Delta_{\mu,\lambda}\mathbf{u} - \nabla(\lambda^*(x)\mathrm{div}\mathbf{u}_t) + \varrho_1(x)\nabla\theta = 0 & \text{in } Q \equiv \Omega \times (0,T), \\ \theta_t - \Delta\theta + \varrho_2(x)\mathrm{div}\mathbf{u}_t = g & \text{in } Q, \\ \mathbf{u}(x,t) = 0, \quad \theta(x,t) = 0 & \text{on } \Sigma \equiv \Gamma \times (0,T), \end{cases} \tag{1}$$

where the $\cdot_t$ stands for the time derivative, $\nabla = (\partial_1, \partial_2, \partial_3)$, and $\Delta_{\mu,\lambda}$ is the elliptic second order linear differential operator given by

$$\begin{aligned} \Delta_{\mu,\lambda}\mathbf{v}(x) \quad &\equiv \quad \mu\Delta\mathbf{v}(x) + (\mu+\lambda)(\nabla(\mathrm{div}\mathbf{v}(x))) \\ &+ \quad \mathrm{div}\mathbf{v}(x)\nabla\lambda(x) + (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\nabla\mu(x), \quad x \in \Omega, \end{aligned} \tag{2}$$

for $\mathbf{v} = (v_1, v_2, v_3)^T$, where $.^T$ denotes the transpose of matrices. Throughout this paper, $\mathbf{u} = (u_1, u_2, u_3)^T$ denotes the displacement at the location x and the time t, and $\theta \equiv \theta(x,t)$, the temperature, is a scalar function, $g \in H^1(0,T; L^2(\Omega))$ is a heat source. We will assume that the Lamé parameters $\mu, \lambda \in \mathcal{C}^2(\overline{\Omega})$, satisfy

$$\mu(x) \geq \mu_0 > 0, \quad \lambda(x) + 2\mu(x) > 0, \quad \forall x \in \overline{\Omega}.$$

and $\lambda^* \in C^2(\overline{\Omega})$ is the consolidation coefficient which satisfy :

$$\lambda^*(x) \geq k_0 > 0, \qquad x \in \overline{\Omega}.$$

Furthermore, the coupling coefficients ϱ_1, ϱ_2 satisfies :

$$(\varrho_1, \varrho_2) \in (\mathcal{C}^2(\overline{\Omega}))^2; \quad \varrho_1(x) > \varrho_0 > 0 \;\text{ in } \; \text{for all}\, x \in \overline{\Omega}.$$

We assume that the heat source is given by

$$g(x,t) = q(x)k(x,t), \tag{3}$$

where $k \in W^{2,\infty}(Q)$ and $q \in H^2(\Omega)$.

We can prove (e.g., [1, 5]) that the system (1) possesses a unique solution

$$(\mathbf{u}, \theta) \equiv \Big(\mathbf{u}_{(\lambda^*,\varrho_1,\varrho_2)}, \theta_{(\lambda^*,\varrho_1,\varrho_2)}\Big),$$

Let $\omega \subset \Omega$ be a given arbitrarily subdomain such that $\partial\omega \supset \Gamma$, i.e. $\omega = \Omega \cap V$ where V is a neighbourhood of Γ on $\mathbb{R}^3$ and let k and $t_0 \in (0,T)$ be appropriately given.

Inverse Problem : Determine $q(x)$, $x \in \Omega$, by measurements

$$\mathbf{u}\,|_{\omega\times(0,T)}, \quad \mathbf{u}(x,t_0), \quad \text{and} \quad \theta(x,t_0) \qquad x \in \Omega.$$

The main subject of this paper is the inverse problem of determining of g, in the Biot consolidation model in poro-elasticity, uniquely from observed data of displacement vector $\mathbf{u}$ on a suitable subdomain $\omega \subset \Omega$ and the observation data of $\mathbf{u}$ and θ at given a suitable time t_0. Such kinds of observation data are similar to those considered in (e.g. [2], [3], [4]).

The key ingredient in our argument is an L^2-weighted inequality of Carleman type for coupled mixed hyperbolic-parabolic system. We prove a Hölder stability estimate in our inverse problem. We note that the uniqueness in the inverse problem follows directly from the Hölder stability.

1.1. Statement of main result

Let $t_0 \in (0,T)$ and $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ such that

$$\frac{1}{\sqrt{r_0}} \max_{x\in\overline{\Omega}} |x - x_0| < \min\{t_0, T - t_0\}, \tag{4}$$

where $r_0 \in (0, \mu_0)$.

We denote $(\mathbf{u}, \theta)$ the solution of (1) corresponding to $(\varrho_1, \varrho, \lambda^*)$.

Theorem 1.1 *Let $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $t_0 \in (0,T)$ satisfies (4).*
Let $k \in W^{2,\infty}(Q)$ such that $\|k\|_{W^{2,\infty}} \leq M$ and

$$k(x, t_0) \neq 0, \qquad x \in \overline{\Omega}.$$

We assume that the solution $(\mathbf{u}, \theta)$ satisfies the a priori boundedeness :

$$\|\mathbf{u}\|_{H^2(0,T;H^2(\Omega))} + \|\theta\|_{H^1(0,T;L^2(\Omega))} \leq M_0, \tag{5}$$

for some given positive constant M_0.
Then there exist constants $C > 0$ and $\delta \in (0,1)$ such that the following stability estimate hold

$$\|q(x)\|_{L^2(\Omega)} \leq C(\|\mathbf{u}_t\|_{H^4(\omega\times(0,T))} + \|\theta(.,t_0)\|_{H^2(\Omega)} + \|\mathbf{u}_t(.,t_0)\|_{H^3(\Omega)} + \|\mathbf{u}_{tt}(.,t_0)\|_{H^1(\Omega)})^{\delta}.$$

By Theorem 1.1, we can readily derive the uniqueness in the inverse problem

Corollary 1.2 *Under the same assumptions as in Theorem 1.1, we have the uniqueness.*
Let $(\mathbf{u}, \theta)$ satisfy the Biot system (1) such that

$$\mathbf{u}(x,t) = 0, \qquad (x,t) \in \omega \times (0,T)$$

and

$$\mathbf{u}(x, t_0) = 0, \quad \theta(x, t_0) = 0, \qquad x \in \Omega.$$

Then

$$q(x) = 0 \ \text{for all } x \in \Omega.$$

The remainder of the paper is organized as follows. In section 2, we give a Carleman estimate for a coupled hyperbolic-parabolic system. In section 3, we prove Theorem 1.1.

2. Carleman estimate

Here we present a Carleman estimate, which was proved in [3]. For formulating our Carleman estimate, we introduce some notations. Let $\psi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by setting

$$\psi(x,t) = \vartheta(x) - \beta((t-t_0)^2 - M) = |x - x_0|^2 - \beta((t-t_0)^2 - M), \quad x \in \overline{\Omega}, \quad (6)$$

for $M > 0$ large, where $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $t_0 \in (0,T)$ such that

$$\min\{t_0^2, (T-t_0)^2\} > r_0^{-1} \left(\max_{x \in \overline{\Omega}} \vartheta(x) \right). \quad (7)$$

We fix $\delta > 0$ and $\beta > 0$ satisfying

$$\beta \min\{t_0^2, (T-t_0)^2\} > \max_{x \in \overline{\Omega}} \vartheta(x) + \delta, \quad 0 < \beta < r_0. \quad (8)$$

Therefore, by (6) and (8) the function $\psi(x,t)$ verifies the following properties

$$\psi(x,0) \leq \beta M - \delta, \quad \psi(x,T) \leq \beta M - \delta, \quad \text{for all} \quad x \in \overline{\Omega}, \quad (9)$$

there exists $\epsilon \in (0, T/4)$ such that

$$\max_{x \in \overline{\Omega}} \psi(x,t) \leq \beta M - \frac{\delta}{2}, \quad \text{for all} \quad t \in (0, 2\epsilon) \cup (T - 2\epsilon, T), \quad (10)$$

and

$$\min_{x \in \overline{\Omega}} \psi(x, t_0) \geq \beta M. \quad (11)$$

We next introduce a function $\varphi : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\varphi(x,t) = e^{\gamma \psi(x,t)} := \rho(x)\alpha(t), \quad \gamma > 0, \quad (12)$$

where γ is a large parameter, $\rho(x)$ and $\alpha(t)$ are defined by

$$\rho(x) = e^{\gamma(|x-x_0|^2 + \beta M)} \geq e^{\gamma \beta M} \equiv d_1, \forall x \in \Omega \text{ and } \alpha(t) = e^{-\beta\gamma(t-t_0)^2} \leq 1, \forall t \in (0,T) \quad (13)$$

and let

$$\sigma \equiv \sigma(x,t) = s\gamma\varphi(x,t). \quad (14)$$

We use usual function spaces, $H^k(Q)$, and

$$H^{2,1}(Q) = H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

Let $(\mathbf{v}, y)$ a solution of the linear Biot consolidation system

$$\left\{ \begin{array}{ll} \mathbf{v}_{tt}(x,t) - \Delta_{\mu,\lambda}\mathbf{v}(x,t) - \nabla(\lambda^* \mathrm{div}\mathbf{v}_t(x,t)) + \varrho_1 \nabla y(x,t) = \mathbf{f}(x,t) & \text{in } Q, \\ y_t(x,t) - \Delta y(x,t) + \varrho_2 \mathrm{div}\mathbf{v}_t(x,t) = h(x,t) & \text{in } Q, \end{array} \right. \quad (15)$$

such that

$$\begin{array}{lll} \mathrm{Supp}(\mathbf{v}(\cdot,t)) \subset \Omega, \quad \mathrm{Supp}(y(\cdot,t)) \subset \Omega, & \text{for all} & t \in (0,T). \\ \partial_t^j \mathbf{v}(x,0) = \partial_t^j \mathbf{v}(x,T) = 0, \quad y(x,0) = y(x,T) = 0 & \text{for all} & x \in \Omega, \ j = 0, 1. \end{array} \quad (16)$$

The following theorem is a weighted Carleman estimate with second large parameter for Biot's consolidation system (15) with assumption (16).

Theorem 2.1 (Carleman estimate) *There exist two constants $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$, there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds*

$$\begin{aligned} & C\int_Q \Big(\sigma\left|\nabla_{x,t}\mathbf{v}\right|^2+\sigma^3\left|\mathbf{v}\right|^2+\sigma^4\left|\mathrm{div}\mathbf{v}\right|^2+\sigma^3\left|\mathrm{div}\mathbf{v}_t\right|^2+\sigma^2\left|\nabla\mathrm{div}\mathbf{v}\right|^2+\sigma\left|\nabla\mathrm{div}\mathbf{v}_t\right|^2 \\ & \quad +\left|\Delta y\right|^2+\sigma^2\left|\nabla y\right|^2+\sigma^4\left|y\right|^2\Big)e^{2s\varphi}dx\,dt \\ \leq & \int_Q (\left|\mathbf{f}\right|^2+\left|\nabla\mathbf{f}\right|^2+\gamma^{-1}\sigma\left|h\right|^2)e^{2s\varphi}dx\,dt \\ & +\int_Q \Big(\gamma^{-1}\left|\Delta\mathrm{div}\mathbf{v}(x,t_0)\right|^2+\gamma^{-1}\left|\mathrm{div}\mathbf{v}_t(x,t_0)\right|^2 \\ & \quad +\sigma^4\left|\mathrm{div}\mathbf{v}(x,t_0)\right|^2+\sigma^2\left|\nabla\mathrm{div}\mathbf{v}(x,t_0)\right|^2\Big)e^{2s\varphi}dxdt, \end{aligned} \tag{17}$$

for any solution $(\mathbf{v}, y) \in H^2(Q) \times H^{2,1}(Q)$ to problem (15) which satisfy (16) and any $s \geq s_*$.

The proof is given by Bellassoued and Riahi [3].

3. Proof of the main result

This section is devoted to the proof of the Theorem 1.1. The idea of the proof is based on the Carleman estimate method in [3]. A usual methodology by [2, 3].

3.1. Preliminaries estimate

Lemma 3.1 *Let ω be an open subdomain of Ω with regular boundary $\partial\omega \supset \Gamma$. There exists constants γ_*, s_* and $C > 0$ such that for any $s \geq s_*$ and any $\gamma \geq \gamma_*$ the following estimate holds :*

$$\int_{\omega\times(0,T)} \sigma^4\left|v\right|^2 e^{2s\varphi}dxdt \leq C\int_{\omega\times(0,T)} \sigma^2\left|\nabla v\right|^2 e^{2s\varphi}dxdt \tag{18}$$

for any $v \in H^1(\omega \times (0, T))$ such that $v(x, t) = 0$ on $\partial\omega \times (0, T)$.

Proof We multiply ∇v by $(\nabla\varphi)ve^{2s\varphi}$ and using the divergence theorem, we obtain

$$\begin{aligned} \int_\omega \nabla v\cdot(\nabla\varphi)v\, e^{2s\varphi}dx &= -\int_\omega v\,\mathrm{div}((\nabla\varphi)\,v\, e^{2s\varphi})dx \\ &= -\int_\omega \left|v\right|^2\Delta\varphi\, e^{2s\varphi}dx-2s\int_\omega\left|v\right|^2\left|\nabla\varphi\right|^2 e^{2s\varphi}dx \\ &\quad -\int_\omega \nabla v\cdot(\nabla\varphi)\,v\, e^{2s\varphi}dx. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\int_\omega \sigma\nabla v\cdot(\nabla\vartheta)\,v\, e^{2s\varphi}dx &= -2\int_\omega \sigma^2\left|v\right|^2\left|\nabla\vartheta\right|^2 e^{2s\varphi}dx \\ &\quad -\int_\omega \sigma\left|v\right|^2\Delta\vartheta\, e^{2s\varphi}dx-\gamma\int_\omega\sigma\left|v\right|^2\left|\nabla\vartheta\right|^2 e^{2s\varphi}dx. \end{aligned}$$

Taking $\gamma \geq \gamma_*$ and $s \geq s_*$ sufficiently large, we obtain for any $\varepsilon > 0$

$$C\int_\omega \sigma^2\left|v\right|^2 e^{2s\varphi}dx \leq C_\varepsilon\int_\omega\left|\nabla v\right|^2 e^{2s\varphi}dx+\varepsilon\int_\omega\sigma^2\left|v\right|^2 e^{2s\varphi}dx. \tag{19}$$

Integrating in $(0,T)$ and taking ε small we obtain

$$C\int_{\omega\times(0,T)}\sigma^2\,|v|^2\,e^{2s\varphi}dx\leq C\int_{\omega\times(0,T)}|\nabla v|^2\,e^{2s\varphi}dx. \tag{20}$$

Applying the last inequality to $\sigma^2 v$ we obtain

$$\int_{\omega\times(0,T)}\sigma^4\,|v|^2\,e^{2s\varphi}dx\leq C\int_{\omega\times(0,T)}\sigma^2\,|\nabla v|^2\,e^{2s\varphi}dx, \tag{21}$$

for any $\gamma\geq\gamma_*$ and $s\geq s_*$. This completes the proof. Hence, by Lemma 3.1, we obtain the following Lemma.

Lemma 3.2 *Let $(\mathbf{v},y)\in H^2(Q)\times H^{2,1}(Q)$, satisfying*

$$\left\{\begin{array}{ll}\mathbf{v}_{tt}-\Delta_{\mu,\lambda}\mathbf{v}-\nabla(\lambda^*(x)\mathrm{div}\mathbf{v}_t)+\varrho_1(x)\,\nabla y=\mathbf{f} & (x,t)\in Q,\\ y_t-\Delta y+\varrho_2(x)\,\mathrm{div}\mathbf{v}_t=h & (x,t)\in Q,\\ \mathbf{v}=0,\qquad y=0 & (x,t)\in\Sigma,\end{array}\right. \tag{22}$$

and

$$\partial_t^j\mathbf{v}(x,0)=\partial_t^j\mathbf{v}(x,T)=0,\quad y(x,0)=y(x,T)=0\qquad \textit{for all}\quad x\in\Omega,\ j=0,1. \tag{23}$$

There exist positive constants γ_* and $C>0$ such that, for any $\gamma\geq\gamma_*$ we can find s_* and D, the following inequality holds

$$\begin{aligned}&\int_Q e^{2s\varphi}\Big(\sigma\,|\nabla_{x,t}\mathbf{v}|^2+\sigma^3\,|\mathbf{v}|^2+\sigma^4\,|\mathrm{div}\mathbf{v}|^2+\sigma^3\,|\mathrm{div}\mathbf{v}_t|^2+\sigma^2\,|\nabla\mathrm{div}\mathbf{v}|^2\\ &+\sigma\,|\nabla\mathrm{div}\mathbf{v}_t|^2+|\Delta y|^2+\sigma^2\,|\nabla y|^2+\sigma^4\,|y|^2\Big)dxdt\leq C\int_Q\Big(\gamma^{-1}\sigma\,|h|^2\\ &\quad+\sigma^2\,|\mathbf{f}|^2+|\nabla\mathbf{f}|^2\Big)dxdt+Ce^{Ds}\Big(\,\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))}\\ &\qquad+\|\mathbf{v}(\cdot,t_0)\|^2_{H^3(\Omega)}+\|\mathbf{v}_t(\cdot,t_0)\|^2_{H^1(\Omega)}\Big),\end{aligned} \tag{24}$$

for any $s\geq s_*$.

Proof Let $\omega'\subset\omega$ such that $\partial\omega'\supset\Gamma$. In order to apply Carleman estimate, we introduce a cut-off function χ satisfying $0\leq\chi\leq 1$, $\chi\in\mathscr{C}^\infty(\mathbb{R}^3)$, $\chi=1$ in $\overline{\Omega\backslash\omega'}$ and supp$\chi\subset\Omega$. Put

$$\widehat{\mathbf{v}}(x,t)=\chi(x)\mathbf{v}(x,t),\qquad \widehat{y}(x,t)=\chi(x)y(x,t).$$

Noting that $(\widehat{\mathbf{v}},\widehat{y})\in H^2(Q)\times H^{2,1}(Q)$, and satisfying

$$\left\{\begin{array}{ll}\widehat{\mathbf{v}}_{tt}-\Delta_{\mu,\lambda}\widehat{\mathbf{v}}-\nabla(\lambda^*\mathrm{div}\widehat{\mathbf{v}}_t)+\varrho_1\nabla\widehat{y}=\widehat{\mathbf{f}} & (x,t)\in Q,\\ \widehat{y}_t-\Delta\widehat{y}+\varrho_2\mathrm{div}\widehat{\mathbf{v}}_t=\widehat{g} & (x,t)\in Q,\end{array}\right. \tag{25}$$

with

$$\mathrm{Supp}(\widehat{\mathbf{v}}(\cdot,t))\subset\Omega\quad \mathrm{Supp}(\widehat{y}(\cdot,t))\subset\Omega,\quad \forall\,t\in(0,T).$$

Here

$$\begin{array}{rcl}\widehat{\mathbf{f}}&=&\chi(x)\mathbf{f}(x,t)-[\Delta_{\mu,\lambda},\chi]\mathbf{v}-\nabla(\lambda^*\nabla\chi.\mathbf{v}_t)-\lambda^*(\nabla\chi)\mathrm{div}\mathbf{v}_t+\varrho_1 y\nabla\chi,\\ &\equiv&\chi(x)\mathbf{f}(x,t)+P_1\mathbf{v}+Q_1\mathbf{v}_t+A_0 y\end{array}$$

$$\begin{aligned}\widehat{g} &= \chi(x)h(x,t) - 2\nabla\chi\cdot\nabla y - y\Delta\chi + \varrho_2\nabla\chi\cdot\mathbf{v}_t \\ &\equiv \chi(x)h(x,t) + A_1 y + Q_0\mathbf{v}_t\end{aligned}$$

and P_1, Q_1, A_1 are a first order partial differential operators with the coefficients are supported in ω' and A_0, Q_0 are zeroth order partial diffrential operators supported also in ω'. Noting that $(\widehat{\mathbf{v}}, \widehat{y})$ satisfies (25), then we can apply the Carleman estimate for Biot's system (17) to $(\widehat{\mathbf{v}}, \widehat{y})$, we obtain

$$\begin{aligned}&C\int_{(\Omega\backslash\omega')\times(0,T)} e^{2s\varphi}\Big(\sigma\left|\nabla_{x,t}\mathbf{v}\right|^2 + \sigma^3\left|\mathbf{v}\right|^2 + \sigma^4\left|\mathrm{div}\mathbf{v}\right|^2 + \sigma^3\left|\mathrm{div}\mathbf{v}_t\right|^2 \\ &+\sigma^2\left|\nabla\mathrm{div}\mathbf{v}\right|^2 + \sigma\left|\nabla\mathrm{div}\mathbf{v}_t\right|^2 + \left|\Delta y\right|^2 + \sigma^2\left|\nabla y\right|^2 + \sigma^4\left|y\right|^2\Big)dxdt \\ &\le \int_Q e^{2s\varphi}\Big(\gamma^{-1}\sigma\left|\widehat{g}\right|^2 + |\widehat{\mathbf{f}}|^2 + |\nabla\widehat{\mathbf{f}}|^2\Big)dxdt + \int_Q\Big(\gamma^{-1}\left|\Delta\mathrm{div}\widehat{\mathbf{v}}(x,t_0)\right|^2 \\ &+\gamma^{-1}\left|\mathrm{div}\widehat{\mathbf{v}}_t(x,t_0)\right|^2 + \sigma^4\left|\mathrm{div}\widehat{\mathbf{v}}(x,t_0)\right|^2 + \sigma^2\left|\nabla\mathrm{div}\widehat{\mathbf{v}}(x,t_0)\right|^2\Big)e^{2s\varphi}dxdt. \quad (26)\end{aligned}$$

Using the first equation of system (22) after taking divergence, we obtain the following estimate

$$\begin{aligned}\int_{\omega'\times(0,T)}\left|\Delta y\right|^2 e^{2s\varphi}dxdt &\le Ce^{Ds}\left\|\mathbf{v}\right\|^2_{H^4(\omega\times(0,T))} \\ &+\int_{\omega'\times(0,T)}\left|\nabla y\right|^2 e^{2s\varphi}dxdt + \int_Q e^{2s\varphi}|\nabla\mathbf{f}|^2dxdt.\end{aligned}$$

By the last inequality and (26), we deduce

$$\begin{aligned}&C\int_{\Omega\times(0,T)} e^{2s\varphi}\Big(\sigma\left|\nabla_{x,t}\mathbf{v}\right|^2 + \sigma^3\left|\mathbf{v}\right|^2 + \sigma^4\left|\mathrm{div}\mathbf{v}\right|^2 + \sigma^3\left|\mathrm{div}\mathbf{v}_t\right|^2 + \sigma^2\left|\nabla\mathrm{div}\mathbf{v}\right|^2 \\ &\quad+\sigma\left|\nabla\mathrm{div}\mathbf{v}_t\right|^2 + \left|\Delta y\right|^2 + \sigma^2\left|\nabla y\right|^2 + \sigma^4\left|y\right|^2\Big)dxdt \\ &\le C\int_Q\Big(\gamma^{-1}\sigma\left|h\right|^2 + \left|\mathbf{f}\right|^2 + \left|\nabla\mathbf{f}\right|^2\Big)dxdt \\ &\quad+C\Big(\int_{\omega'\times(0,T)}\sigma^4\left|y\right|^2 e^{2s\varphi}dxdt + \int_{\omega'\times(0,T)}\sigma^2\left|\nabla y\right|^2 e^{2s\varphi}dxdt\Big) \\ &\quad+e^{Ds}\Big(\left\|\mathbf{v}\right\|^2_{H^4(\omega\times(0,T))} + \left\|\mathbf{v}(\cdot,t_0)\right\|^2_{H^3(\Omega)} + \left\|\mathbf{v}_t(\cdot,t_0)\right\|^2_{H^1(\Omega)}\Big). \quad (27)\end{aligned}$$

Let χ_1 be a cut-off function satisfying $0 \le \chi_1 \le 1$, $\chi_1 \in \mathscr{C}^\infty(\mathbb{R}^3)$, $\chi_1 = 1$ in $\overline{\omega'}$ and $\mathrm{Supp}(\chi_1) \subset \omega$. Let us consider $z(x,t) = \chi_1(x)y(x,t) \in H^1(\omega\times(0,T))$ and $z(x,t) = 0$ for all $(x,t) \in \partial\omega\times(0,T)$, so that by Lemma 3.1, we have

$$\begin{aligned}&\int_{\omega'\times(0,T)}\sigma^4\left|y\right|^2\sigma e^{2s\varphi}dxdt \le \int_{\omega\times(0,T)}\sigma^4\left|z\right|^2 e^{2s\varphi}dxdt \\ &\le C\int_{\omega\times(0,T)}\sigma^2\left|\nabla y\right|^2 e^{2s\varphi}dxdt + C\int_Q\sigma^2\left|y\right|^2 e^{2s\varphi}dxdt. \quad (28)\end{aligned}$$

Furthermore by the first equation of (22), we have

$$\int_{\omega'\times(0,T)}\sigma^2\left|\nabla y\right|^2 e^{2s\varphi}dxdt \le Ce^{Ds}\left\|\mathbf{v}\right\|^2_{H^3(\omega\times(0,T))}$$

$$+\int_{\Omega\times(0,T)}\sigma^2|\mathbf{f}|^2 e^{2s\varphi}dxdt. \tag{29}$$

Inserting (28) and (29) in (27), we obtain (24). This completes the proof of the Lemma.

Let $(\mathbf{v},y)\in H^2(Q)\times H^{2,1}(Q)$ satisfying

$$\begin{array}{ll}\mathbf{v}_{tt}-\Delta_{\mu,\lambda}\mathbf{v}-\nabla(\lambda^*\mathrm{div}\mathbf{v}_t)+\varrho_1\nabla y=\mathbf{f} & (x,t)\in Q,\\ y_t-\Delta y+\varrho_2\,\mathrm{div}\,\mathbf{v}_t=h & (x,t)\in Q,\\ \mathbf{v}=0,\ y=0 & (x,t)\in\Sigma.\end{array}$$

Put

$$\widetilde{\mathbf{v}}(x,t)=\eta(t)\mathbf{v}(x,t),\qquad \widetilde{y}(x,t)=\eta(t)y(x,t).$$

Noting that $(\widetilde{\mathbf{v}},\widetilde{y})\in H^2(Q)\times H^{2,1}(Q)$ satisfies

$$\begin{array}{ll}\widetilde{\mathbf{v}}_{tt}-\Delta_{\mu,\lambda}\widetilde{\mathbf{v}}-\nabla(\lambda^*\mathrm{div}\widetilde{\mathbf{v}}_t)+\varrho_1\nabla\widetilde{y}=\eta\mathbf{f}+\eta_{tt}\mathbf{v}+2\eta_t\mathbf{v}_t-\eta_t\nabla(\lambda^*\mathrm{div}\mathbf{v}) & (x,t)\in Q,\\ \widetilde{y}_t-\Delta\widetilde{y}+\varrho_2\,\mathrm{div}\,\widetilde{\mathbf{v}}_t=\eta h+\eta_t(y-\varrho_2\mathrm{div}\mathbf{v}) & (x,t)\in Q,\\ \widetilde{\mathbf{v}}=0,\ \widetilde{y}=0 & (x,t)\in\Sigma,\end{array} \tag{30}$$

Henceforth we fix $\gamma>0$ sufficiently large. By $N_{s,\varphi}$ we denote the quantity

$$N_{s,\varphi}(\mathbf{v},y)=\int_Q e^{2s\varphi}\Big(s|\nabla_{x,t}\mathbf{v}|^2+s^3|\mathbf{v}|^2+s^4|\mathrm{div}\mathbf{v}|^2+s^3|\mathrm{div}\mathbf{v}_t|^2$$
$$+s^2|\nabla\mathrm{div}\mathbf{v}|^2+s|\nabla\mathrm{div}\mathbf{v}_t|^2+|\Delta y|^2+s^2|\nabla y|^2+s^4|y|^2\Big)dxdt. \tag{31}$$

We introduce a cut-off function η satisfying $0\le\eta\le 1$, $\eta\in\mathcal{C}^\infty(\mathbb{R})$, $\eta=1$ in $(2\varepsilon,T-2\varepsilon)$ and
$\mathrm{Supp}(\eta)\subset(\varepsilon,T-\varepsilon)$. Finally we denote

$$\widetilde{\mathbf{v}}=\eta\mathbf{v},\qquad \widetilde{y}=\eta y.$$

Setting $d_0=e^{(\beta M-\delta/2)\gamma}$, by (9), we have

$$\max_{x\in\Omega}\varphi(x,t)\le d_0,\quad t\in(0,2\varepsilon)\cup(T-2\varepsilon,T). \tag{32}$$

We have the following lemma :

Lemma 3.3 *There exist three positive constants s_*, $C>0$ and D such that the following inequality holds :*

$$CN_{s,\varphi}(\widetilde{\mathbf{v}},\widetilde{y})\le\int_Q\Big(s|h|^2+s^2|\mathbf{f}|^2+|\nabla\mathbf{f}|^2\Big)e^{2s\varphi}dxdt$$
$$+e^{Ds}\Big(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))}+\|\mathbf{v}(\cdot,t_0)\|^2_{H^3(\Omega)}+\|\mathbf{v}_t(\cdot,t_0)\|^2_{H^1(\Omega)}\Big)$$
$$+Cs^2e^{2d_0s}\Big(\|\mathbf{v}\|^2_{H^1(0,T;H^2(\Omega))}+\|y\|^2_{L^2(Q)}\Big),$$

for any $s\ge s_*$ and any $(\widetilde{\mathbf{v}},\widetilde{y})\in H^2(Q)\times H^{2,1}(Q)$ satisfying (30).

Proof Applying Carleman estimate (24) to $(\widetilde{\mathbf{v}},\widetilde{y})$, we obtain

$$CN_{s,\varphi}(\widetilde{\mathbf{v}},\widetilde{y}) \leq \int_Q \left(s\,|h|^2 + s^2\,|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2\right) e^{2s\varphi} dxdt$$
$$+ e^{Ds}\left(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))} + \|\mathbf{v}(\cdot,t_0)\|^2_{H^3(\Omega)} + \|\mathbf{v}_t(\cdot,t_0)\|^2_{H^1(\Omega)}\right)$$
$$+ \int_Q (s^2(|\eta_{tt}|^2 + |\eta_t|^2)(|y|^2 + |\mathbf{v}|^2 + |\mathbf{v}_t|^2 + |\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}_t|^2 + |\nabla(\mathrm{div}\mathbf{v})|^2) e^{2s\varphi} dxdt,$$

for any $\gamma \geq \gamma_*$ and $s \geq s_*$. Since $\mathrm{Supp}(\eta_{tt}),\ \mathrm{Supp}(\eta_t) \subset (0,2\epsilon)\cup(T-2\epsilon,T)$, we obtain from (32)

$$\int_Q (s^2(|\eta_{tt}|^2 + |\eta_t|^2)(|y|^2 + |\mathbf{v}|^2 + |\mathbf{v}_t|^2 + |\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}_t|^2 + |\nabla(\mathrm{div}\mathbf{v})|^2) e^{2s\varphi} dxdt$$
$$\leq Cs^2 e^{2d_0 s}\left(\|\mathbf{v}\|^2_{H^1(0,T;H^2(\Omega))} + \|y\|^2_{L^2(Q)}\right).$$

This completes the proof of the lemma.

3.2. Completion of the proof of the main result

Consider now the following system

$$\begin{array}{ll} \mathbf{u}_{tt} - \Delta_{\mu,\lambda}\mathbf{u} - \nabla(\lambda^*\mathrm{div}\mathbf{u}_t) + \varrho_1\nabla\theta = 0 & (x,t)\in Q, \\ \theta_t - \Delta\theta + \varrho_2\,\mathrm{div}\,\mathbf{u}_t = g & (x,t)\in Q, \\ \mathbf{u} = 0,\ \theta = 0 & (x,t)\in\Sigma, \end{array} \tag{33}$$

where the heat source term g is given by

$$g(x,t) = q(x)k(x,t).$$

Let $\mathbf{v} = u_t$ and $y = \theta_t$. Then, we have

$$\begin{array}{ll} \mathbf{v}_{tt} - \Delta_{\mu,\lambda}\mathbf{v} - \nabla(\lambda^*\mathrm{div}\mathbf{v}_t) + \varrho_1\nabla y = 0 & (x,t)\in Q, \\ y_t - \Delta y + \varrho_2\,\mathrm{div}\,\mathbf{v}_t = g_t & (x,t)\in Q, \\ \mathbf{v} = 0,\ y = 0 & (x,t)\in\Sigma. \end{array} \tag{34}$$

We apply Lemma 3.3 to $(\widetilde{\mathbf{v}},\widetilde{y})$ solution of the following system

$$\begin{array}{ll} \widetilde{\mathbf{v}}_{tt} - \Delta_{\mu,\lambda}\widetilde{\mathbf{v}} - \nabla(\lambda^*\mathrm{div}\widetilde{\mathbf{v}}_t) + \varrho_1\nabla\widetilde{y} = \eta_{tt}\mathbf{v} + 2\eta_t\mathbf{v}_t - \eta_t\nabla(\lambda^*\mathrm{div}\mathbf{v}) & (x,t)\in Q, \\ \widetilde{y}_t - \Delta\widetilde{y} + \varrho_2\,\mathrm{div}\,\widetilde{\mathbf{v}}_t = \eta g_t + \eta_t(y - \varrho_2\mathrm{div}\mathbf{v}) & (x,t)\in Q, \\ \widetilde{\mathbf{v}} = 0,\ \widetilde{y} = 0 & (x,t)\in\Sigma, \end{array} \tag{35}$$

we obtain the following estimate :

$$\begin{aligned} CN_{s,\varphi}(\widetilde{\mathbf{v}},\widetilde{y}) &\leq \int_Q s\,|g_t|^2 e^{2s\varphi} dxdt \\ &+ e^{Ds}\left(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))} + \|\mathbf{v}(\cdot,t_0)\|^2_{H^3(\Omega)} + \|\mathbf{v}_t(\cdot,t_0)\|^2_{H^1(\Omega)}\right) \\ &+ Cs^2 e^{2d_0 s}\left(\|\mathbf{v}\|^2_{H^1(0,T;H^2(\Omega))} + \|y\|^2_{L^2(Q)}\right), \end{aligned} \tag{36}$$

for sufficiently large $s > 0$.
We have the following Lemma

Lemma 3.4 *There exists a positive constant $C > 0$ such that the following estimate*

$$\int_\Omega |z(x,t_0)|^2\,dx \le C\int_Q (s\,|z(x,t)|^2 + s^{-1}\,|z_t(x,t)|^2)dxdt,$$

for any $z \in H^1(0,T;L^2(\Omega))$.

Proof By direct computations, we have

$$\begin{aligned}\int_\Omega \eta^2(t_0)\,|z(x,t_0)|^2\,dx &= \int_0^{t_0}\frac{d}{dt}\Big(\int_\Omega \eta^2(t)\,|z(x,t)|^2\,dx\Big)dt\\ &= 2\int_0^{t_0}\int_\Omega \eta^2(t)z(x,t)z_t(x,t)dxdt + 2\int_0^{t_0}\int_\Omega \eta_t(t)\eta(t)\,|z(x,t)|^2\,dxdt.\end{aligned}$$

Then, we have

$$\int_\Omega |z(x,t_0)|^2\,dx \le C\int_Q (s\,|z(x,t)|^2 + s^{-1}\,|z_t(x,t)|^2)dxdt.$$

This complete the proof of the lemma.

Second, we apply lemma 3.4 to $y_1(x,t) = \eta(t)e^{2s\varphi(x,t)}y(x,t) = e^{2s\varphi(x,t)}\widetilde{y}(x,t)$ and by the second equation of (35), we obtain

$$\begin{aligned}&\int_\Omega e^{2s\rho(x)}\,|y(x,t_0)|^2\,dx \le \int_Q \sigma e^{2s\varphi}\,|\widetilde{y}(x,t)|^2\,dxdt + \int_Q \sigma^{-1}e^{2s\varphi}\,|\widetilde{y}_t(x,t)|^2\,dxdt\\ &\le \int_Q \sigma e^{2s\varphi}\,|\widetilde{y}(x,t)|^2\,dxdt + \int_Q \sigma^{-1}e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt\\ &+\int_Q \sigma^{-1}e^{2s\varphi}\,|\Delta\widetilde{y}(x,t)|^2\,dxdt + \int_Q \sigma^{-1}e^{2s\varphi}\,|\mathrm{div}\widetilde{\mathbf{v}}_t|^2\,dxdt\\ &+\int_Q \sigma^{-1}e^{2s\varphi}\,|\eta_t|^2\,(|y|^2 + |\mathrm{div}\mathbf{v}|^2)dxdt\\ &\le \int_Q \sigma e^{2s\varphi}\,|\widetilde{y}(x,t)|^2\,dxdt + \int_Q \sigma^{-1}e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt\\ &+\int_Q \sigma^{-1}e^{2s\varphi}\,|\Delta\widetilde{y}(x,t)|^2\,dxdt + \int_Q \sigma^{-1}e^{2s\varphi}\,|\mathrm{div}\widetilde{\mathbf{v}}_t|^2\,dxdt\\ &+e^{2sd_0}\Big(\,\|\mathbf{v}\|^2_{H^1(Q)} + \|y\|^2_{L^2(Q)}\Big).\end{aligned}\tag{37}$$

Then,

$$\begin{aligned}&Cs\int_\Omega e^{s\rho(x)}\,|y(x,t_0)|^2\,dx \le \int_Q e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt\\ &+\int_Q e^{2s\varphi}\Big(|\Delta\widetilde{y}|^2 + \sigma^2\,|\widetilde{y}|^2 + |\mathrm{div}\widetilde{\mathbf{v}}_t|^2\Big)dxdt + \ s\ e^{2sd_0}\ (\|\mathbf{v}\|^2_{H^1(Q)} + \|y\|^2_{L^2(Q)})\\ &\le \int_Q e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt + C\mathcal{N}_{s,\varphi}(\widetilde{\mathbf{v}},\widetilde{y}) + se^{2sd_0}(\|\mathbf{v}\|^2_{H^1(Q)} + \|y\|^2_{L^2(Q)}).\end{aligned}\tag{38}$$

Moreover, using (36), we get

$$s\int_\Omega e^{2s\rho(x)}\,|y(x,t_0)|^2\,dx \le Cs\int_Q e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt$$

$$+e^{Ds}\Big(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))} + \|\mathbf{v}(.,t_0)\|^2_{H^3(\Omega)} + \|\mathbf{v}_t(.,t_0)\|^2_{H^1(\Omega)} \Big)$$
$$+Cs^2e^{2sd_0}\Big(\|\mathbf{v}\|^2_{H^1(0,T;H^2(\Omega))} + \|y\|^2_{L^2(Q)} \Big). \tag{39}$$

On the one hand, by the second equation in (33), we have

$$y(x,t_0) = \Delta\theta(x,t_0) - \varrho_2 \mathrm{div}\mathbf{v}(x,t_0) + q(x)k(x,t_0). \tag{40}$$

Moreover, we have

$$|q(x)| \le |q(x)|\,|k(x,t_0)| \le |y(x,t_0)| + |\Delta\theta(x,t_0)| + |\mathrm{div}\mathbf{v}(x,t_0)|\,. \tag{41}$$

Then,

$$\int_\Omega e^{2s\rho(x)}\,|q(x)|^2\,dx \le \int_\Omega e^{2s\rho(x)}\,|y(x,t_0)|^2\,dx$$
$$+\int_\Omega e^{2s\rho(x)}\,|\mathrm{div}\mathbf{v}(x,t_0)|^2\,dx + \int_\Omega e^{2s\rho(x)}\,|\Delta\theta(x,t_0)|^2\,dx. \tag{42}$$

We deduce that

$$s\int_\Omega e^{2s\rho(x)}\,|q(x)|^2\,dx \le s\int_\Omega e^{2s\rho(x)}\,|y(x,t_0)|^2\,dx$$
$$+s\int_\Omega e^{2s\rho(x)}\,|\mathrm{div}\mathbf{v}(x,t_0)|^2\,dx + Ce^{sD}\,\|\Delta\theta(x,t_0)\|^2_{L^2(\Omega)}\,. \tag{43}$$

Inserting (39) into (43), we get

$$s\int_\Omega e^{2s\rho(x)}\,|q(x)|^2\,dx \le Cs\int_Q e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt + Cs^2e^{2sd_0}M_0^2$$
$$+Ce^{Ds}\Big(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))} + \|\mathbf{v}(.,t_0)\|^2_{H^3(\Omega)}$$
$$+\|\mathbf{v}_t(.,t_0)\|^2_{H^1(\Omega)} + \|\Delta\theta(x,t_0)\|^2_{L^2(\Omega)} \Big). \tag{44}$$

Selecting $\kappa \in (d_0, d_1)$ such that $Cs^2e^{2sd_0} \le e^{2\kappa s}$ for any s large, we get

$$s\int_\Omega e^{2s\rho(x)}\,|q(x)|^2\,dx \le Cs\int_Q e^{2s\varphi}\,|g_t(x,t)|^2\,dxdt + e^{2\kappa s}M_0^2$$
$$+Ce^{Ds}\Big(\|\mathbf{v}\|^2_{H^4(\omega\times(0,T))} + \|\mathbf{v}(.,t_0)\|^2_{H^3(\Omega)}$$
$$+\|\mathbf{v}_t(.,t_0)\|^2_{H^1(\Omega)} + \|\theta(.,t_0)\|^2_{H^2(\Omega)} \Big). \tag{45}$$

Now we return to the first integral term in Q on the right-hand side term of (45).

$$\int_Q e^{2s\varphi}\,|q(x)k_t|^2\,dxdt \le \int_\Omega e^{2s\rho(x)}\,|q(x)|^2\Big(\int_0^T e^{-2s(\rho-\varphi)}\,\|k_t(.,t)\|^2_{L^\infty(\Omega)}\,dt\Big)dx. \tag{46}$$

By the Lebesgue theorem, we obtain

$$\int_0^T e^{-2s(\rho-\varphi)}\,\|k_t(.,t)\|^2_{L^\infty(\Omega)}\,dt = \int_0^T e^{-2s\rho(x)(1-\alpha(t))}\,\|k_t(.,t)\|^2_{L^\infty(\Omega)}\,dt$$

$$\leq \int_0^T e^{-2s(1-\alpha(t))} \|k_t(.,t)\|_{L^\infty}^2 \, dt = o(1), \tag{47}$$

as $s \to \infty$. By (45), we have

$$\begin{aligned} & s\|e^{s\rho}q\|_{L^2(\Omega)}^2 \leq o(1)\int_Q se^{2s\rho}|q(x)|^2\,dxdt + e^{2\kappa s}M_0^2 \\ & +Ce^{Ds}\Big(\|\mathbf{v}\|_{H^4(\omega\times(0,T))}^2 + \|\mathbf{v}(.,t_0)\|_{H^3(\Omega)}^2 \\ & +\|\mathbf{v}_t(.,t_0)\|_{H^1(\Omega)}^2 + \|\theta(.,t_0)\|_{H^2(\Omega)}^2\Big). \end{aligned} \tag{48}$$

for all $s \geq s_*$.
On the other hand, using the fact that $\rho(x) \geq d_1$ for all $x \in \Omega$. Then for sufficiently large $s > 0$, we have

$$\begin{aligned} & \|q\|_{L^2(\Omega)}^2 \leq Ce^{\gamma_1 s}(\|\mathbf{v}\|_{H^4(\omega\times(0,T))}^2 + \|\mathbf{v}(.,t_0)\|_{H^3(\Omega)}^2 \\ & +\|\mathbf{v}_t(.,t_0)\|_{H^1(\Omega)}^2 + \|\theta(.,t_0)\|_{H^2(\Omega)}^2) + Ce^{-\gamma_2 s}M_0^2. \end{aligned} \tag{49}$$

Finally, minimizing the right hand side with respect to s, we obtain

$$\|q\|_{L^2(\Omega)} \leq C(\|\mathbf{v}\|_{H^4(\omega\times(0,T))} + \|\theta(.,t_0)\|_{H^2(Q)} + \|\mathbf{v}(.,t_0)\|_{H^3(\Omega)} + \|\mathbf{v}_t(.,t_0)\|_{H^1(\Omega)})^\delta.$$

Thus proof of Theorem 1.1 is completed.

4. Bibliographie

[1] H. BARUCQ, M.MADAUNE-TORT AND P. SAINT-MACARY, « Theoretical aspects of wave propagation for Biot's consolidation problem », *Monografías del Seminario Mathemático García de Galdeano*, vol. 31, 2004.

[2] M.BELLASSOUED, M. YAMAMOTO, « Carleman estimates and inverse heat source problem for thermoelasticity system », *Inverse problems*, vol. 27, 2011.

[3] M.BELLASSOUED, B. RIAHI « Carleman estimate for Biot consolidation system in poro-elasticity and application to inverse problem », *Mathematical Method in the Applied Sciences*, 2016.

[4] B. WU, J. LIU, « Conditional stability and uniqueness for determining two coefficients in a hyperbolic-parabolic system », *Journal*, vol. 27, 2011.

[5] J-L.LIONS, E.MAGENES, « Non-homogenous boundary value problems and applications », *Springer-Verlag*, vol. I and II, 1972.