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Solvability of Mixed Problems With Integral Condition for Singular Parabolic Equations

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RÉSUMÉ.

ABSTRACT. In this paper we proved the existance and uniqness of strong generalized solution of mixed problems wih integral condition for singular parabolic equaions depending on a theorem proved in [1] in which a priori estimaion of the solution for such problems was derived.

MOTS-CLÉS :

KEYWORDS : Priori estimate; mixed problem; Singular parabolic equations; integral boundary conditions.
2000 Mathematics Subject Classification. 35K20, 35K25, 35K30.

1. Introduction

Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [3], Byszewski et al [4] and [5], Gasymov [7], Ionkin [8]-[9], Lazhar [11], and Said-Nadia [12]. The results and the method used here are a further elaboration of those in [2]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [6] and [13].

In [1] the author considered the following mixed problem in the rectangle $Q = (0, l) \times (0, T)$

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{x^m} \frac{\partial}{\partial x}(x^m \frac{\partial u}{\partial x}) = f(x,t),\ m > 0, \tag{1}$$

$$lu = u(x,0)\varphi(x),\ |u(0,t)| < \infty,\ \iint_{\alpha}^{l} x^m u(x,t)dx = 0,\ \alpha > 0. \tag{2}$$

and he proved the following theorem

Theorem 1 : *For any function $u \in E$ such that $x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q)$ and $x^{-\frac{m}{2}} \frac{\partial}{\partial x} x^m \frac{\partial u}{\partial x} \in L_2(Q)$, the following inequality holds*

$$\|u\|_E^2 \leq c \|\mathcal{F}\|_F^2 , \tag{3}$$

where $c = 2\left(l + \exp(\frac{T}{2\alpha^2})\right)$.

2. The Main Result

we consider the operator $L = (\mathcal{L}, l)$ with the following domain

$$D(L) = \left\{ u \in E : x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q), x^{\frac{m}{2}} \frac{\partial}{\partial x}(x^m \frac{\partial u}{\partial x}) \in L_2(Q) \right\},$$

acting from E into F by the rule $Lu = (\mathcal{L}u, u(x,0))$.

In a standard way its proved [10] that the operator is closable which we denote by $\overline{L}$ with the domain $D(\overline{L})$.

Definition : The solution of the equation $\overline{L}u = \mathcal{F}$ is called strong generalized solution of the problem (1)-(2). In other words, the function u is called strong generalized solution of the problem (1)-(2) if there exist a sequence of functions $u_n \in D(L)$, such that the $\|u_n - u\|_E \to 0$ and $\|Lu_n - \mathcal{F}\|_F \to 0$ at $n \to \infty$.

Theorem 2 : Strong generalized solution of the problem (1)-(2) exist and unique for any $\mathcal{F} = (f, \varphi) \in F$.

Proof. For the sequence $u_n \in D(L)$ the following inequality holds

$$\|u_n\|_E^2 \leq 2\left(l + \exp\left(\frac{T}{2\alpha^2}\right)\right) \|Lu_n\|_F^2 , \tag{4}$$

which implies from theorem 1. Passing in (4) to the limit at $n \to \infty$, we get the following inequality

$$\|u_n\|_E^2 \leq 2\left(l+\exp\left(\frac{T}{2\alpha^2}\right)\right)\left\|\overline{L}u_n\right\|_F^2, u \in D(\overline{L}). \quad (5)$$

From (5) implies that strong generalized solution of problem (1)-(2) is unique, the range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and $R(\overline{L}) = \overline{R(L)}$. Therefore for the proof of existence of strong generalized solution of (1)-(2) we need to prove that the range $R(L)$ of the operator L is dense in F.

Since the range of the operator L is dense in a space with the norm $(\iint\limits_0^l x^n\left(\varphi'\right)^2 dx + \frac{m}{l-\alpha}\iint\limits_\alpha^l x^{m-1}\varphi^2 dx)^{\frac{1}{2}}$, its sufficient to show that the equality

$$\iint\limits_Q x^m \mathcal{L}uvdxdt = 0, \forall u \in D_0(L) = \{u \in D(L) : u(x,0) = 0\}, x^{\frac{m}{2}}v \in L_2(Q), \quad (6)$$

imply the equality $v = 0$.

We set in (6) $x^m v = Mh$, where $Mh = x^m h$ at $0 \leq x \leq \alpha$ and

$$Mh = \frac{l-x}{l-\alpha}x^m h(x,t) + \frac{1}{l-\alpha}\iint\limits_\alpha^x \xi^m h(\xi,t)d\xi, \quad (7)$$

at $\alpha \leq x \leq l$. This holds if the function $h(x,t) = v(x,t)$ at $0 \leq x \leq \alpha$ and

$$h(x,t) = v(x,t) - \frac{1}{(l-\alpha)x^m}\iint\limits_\alpha^l \xi^m v(\xi,t)d\xi,$$

at $\alpha \leq x \leq l$. It is not hard to see that the function h satisfy the third condition from (2), that is

$$\iint\limits_\alpha^l x^m h(x,t)dx = 0. \quad (8)$$

So for any function $u \in D_0(L)$ and given function h we get the equality

$$\iint\limits_Q \frac{\partial u}{\partial t}Mhdxdt = \iint\limits_Q \frac{1}{x^m}\frac{\partial}{\partial x}(x^m\frac{\partial u}{\partial x})Mhdxdt. \quad (9)$$

We set in (9) $u = \iint_0^t w(x,r)dr$ where w is any function such that $x^{\frac{m}{2}}w \in L_2(Q)$,

$$\frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m\frac{\partial w}{\partial x}) \in L_2(Q), \iint\limits_\alpha^l x^m w(x,t)dx = 0, \left|\frac{\partial}{\partial x}w(0,t)\right| < \infty.$$

Then we get the equality

$$\iint\limits_Q wMhdxdt = \iint\limits_Q \frac{1}{x^m}\frac{\partial}{\partial x}(x^m\frac{\partial w}{\partial x})Mgdxdt, \quad (10)$$

where $g(x,t) = \iint_t^T h(x,\tau)d\tau$. The left side of (10) show that the map

$$L_2(Q) \ni x^{\frac{m}{2}} w \to \iint\limits_Q \frac{1}{x^m}\frac{\partial}{\partial x}(x^m \frac{\partial w}{\partial x})Mgdxdt \in R$$

is linear continuos functional. Consequently

$$x^{\frac{m}{2}} \frac{\partial}{\partial x}\frac{Mg}{x^m} \in L_2(Q), \frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m \frac{\partial}{\partial x}(\frac{Mg}{x^m})) \in L_2(Q),$$

and by virtue of (5)

$$x^{\frac{m}{2}} \frac{\partial}{\partial x} g \in L_2(Q), \frac{1}{x^{\frac{m}{2}}}\frac{\partial}{\partial x}(x^m \frac{\partial}{\partial x} g) \in L_2(Q), \left|\frac{\partial g(0,t)}{\partial x}\right| < \infty. \tag{11}$$

Integrating by parts the right hand side of (10) and taking into account (7) and (8), we get

$$\iint\limits_Q wMhdxdt = -\iint\limits_Q x^m \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\frac{Mg}{x^m}dxdt. \tag{12}$$

On the basis of (11) we set in (12)

$$w(x,t) = \iint\limits_0^t e^{c(\tau-T)} g(x,\tau)d\tau. \tag{13}$$

Then

$$\iint\limits_Q e^{c(t-T)} gMgdxdt = -\iint\limits_Q e^{c(\tau-t)} x^m \frac{\partial w}{\partial x}\frac{\partial^2}{\partial x\partial t}(\frac{Mw}{x^m})dxdt. \tag{14}$$

By analogy of formula (12) in [1] we get

$$\begin{aligned} \iint\limits_Q e^{c(t-T)} gMgdxdt &= \int\limits_Q e^{c(t-T)} \psi(x)\left|g\right|^2 dxdt \\ &+ \int\limits_0^T \int\limits_\alpha^l \frac{me^{c(t-T)}}{2(l-\alpha)x^{m+1}}\left|\int\limits_\alpha^x \xi^m g(\xi,t)d\xi\right|^2 dxdt. \end{aligned} \tag{15}$$

Further

$$\begin{aligned}
\int\limits_Q e^{c(T-t)} x^m \frac{\partial w}{\partial x}\frac{\partial^2}{\partial t\partial x}\frac{Mw}{x^m}dxdt &= \int\limits_Q e^{c(T-t)} x^m \psi(x)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}dxdt \\
&- \int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}m}{(l-\alpha)x}\frac{\partial w}{\partial x}\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt \\
&= \frac{1}{2}\int\limits_0^l x^m\psi(x)\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right|^2 dx \\
&+ \frac{c}{2}\int\limits_Q e^{c(T-t)}x^m\psi(x)\left|w(x,t)\right|^2 dxdt \\
&- \int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}m}{(l-\alpha)x^2}w\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt \\
&+ \frac{1}{2}\int\limits_\alpha^l \frac{x^{m-1}m}{(l-\alpha)}\left|\int\limits_0^T e^{c(t-T)}g(x,t)dt\right| dx \\
&+ \frac{cm}{2}\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2dxdt, \qquad (16)
\end{aligned}$$

where $\psi(x) = \begin{cases} 1, & 0 \le x \le \alpha, \\ \frac{l-x}{l-\alpha}, & \alpha \le x \le l. \end{cases}$
By analogy of (18) in [1] we get

$$\begin{aligned}
\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}w}{(l-\alpha)x^2}\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi dxdt &\le \frac{1}{2}\int\limits_0^T\int\limits_0^l \frac{e^{c(T-t)}}{(l-\alpha)x^{m+1}}\left|\int\limits_\alpha^x \xi^m\frac{\partial w}{\partial t}d\xi\right|^2 dxdt \\
&+ \frac{1}{2\alpha^2}\int\limits_0^T\int\limits_\alpha^l \frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2dxdt. \qquad (17)
\end{aligned}$$

From (14) and by virtue of (13),(15)-(17) implies the following inequality

$$\int\limits_{Q} e^{c(t-T)}\psi(x)\left|g\right|^2 dxdt + \frac{1}{2}\int\limits_{0}^{l} x^m\psi(x)\left|\int\limits_{0}^{T} e^{c(t-T)}g(x,t)dt\right|^2 dx$$

$$+\frac{c}{2}\int\limits_{Q} e^{c(T-t)}x^m\psi(x)\left|w(x,t)\right|^2 dxdt + \frac{m}{2}\int\limits_{\alpha}^{l}\frac{x^{m-1}}{(l-\alpha)}\left|\int\limits_{0}^{T} e^{c(t-T)}g(x,t)dt\right| dx$$

$$+\frac{m}{2}(c-\frac{1}{\alpha^2})\int\limits_{0}^{T}\int\limits_{\alpha}^{l}\frac{e^{c(T-t)}x^{m-1}}{(l-\alpha)}w^2(x,t)dxdt \leq 0. \tag{18}$$

We set in (13) $c \geq \frac{1}{\alpha^2}$. Then from (18) implies that $g \equiv 0$. Since $x^m v = M\frac{\partial g}{\partial t}$ then $v \equiv 0$, and theorem 2 is proved.

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