FE approximation for an hybrid Naghdi equations for shells with G_1-midsurface

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RÉSUMÉ. Le but de ce travail est de considérer une formulation hybride de la coque de Naghdi avec une surface moyenne G_1 du modèle déjà introduit par H. Le Dret dans [1] et de prouver sa bonne posture. Ici, le déplacement et la rotation de la normale à la surface moyenne sont donnés respectivement en une base cartésienne et une base locale covariante ou contravariante. Cette nouvelle version nous permet, en particulier, d'approximer la solution par des éléments finis conformes avec moins degrés de liberté. Des tests numériques sont donnés pour illustrer l'eficacité de notre approche.

ABSTRACT. The purpose of this present work is to consider an hybrid formulation of Naghdi's shell with G_1-midsurface of the model already introduced by H. Le Dret in [1] and prove its well-posedness. Here, the displacement and the rotation of the normal to the midsurface are respectively given in Cartesian and local covariant or contravariant basis. This new version enables us, in particular, to approximate by conforming finite elements the solution with less degrees of freedom. Numerical tests are given to illustrate the efficiency of our approach.

MOTS-CLÉS : modèle de coque de Naghdi, surface moyenne G_1, formulation hybride, approximation par éléments finis.

KEYWORDS : Naghdi's shell model, G_1-midsurface, hybrid formulation, finite element approximation.

1. Introduction

The shells and their assemblies are part of a wide variety of elastic structures with big interest for contemporary engineering. Mathematical modeling and numerical analysis of elastic body of three-dimensional problems fail, because of the small thickness parameter relatively small setting [3]. Therefore, it is natural to think about replacing the three-dimensional models by bi-dimensional models posed on the middle surface of the shell for reasons of cost calculation [4]. Following the work of [5, 6, 7, 8], there has been a renewed interest in this subject for the calculation of shells with C^1-midsurface, called shells with little regularity, admitting discontinuities of curvatures.

Among shell models with little regularity developed, we find the Naghdi model with G_1-midsurface introduced in [1]. It is a model which takes into account membrane and flexion deformations as well as the effects of transverse shears for homogeneous and isotropic shells with curvatures discontinuities.

These models are often more described via several charts whose surfaces are made up of contiguous patches and the joins between them are called C_1 conditions or more generally G_1 conditions.

In [15], one can find some examples of G_1 surfaces. For example, to represent a sphere or a torus from a bilipschitzian mapping φ from some domain ω in the hyperplane $\mathbb{R}^2$, we replace ω by a domain with contiguous facets. Each facet will lie in an $\mathbb{R}^2$-dimensional affine subspace of R^2 and it is K-regular patche as defined in [1].

The main result of the present article is to consider mechanical models of shells with such G_1 surfaces by using a formulation without any dual unknown which is inspired from an hybrid formulation introduced to a shell with C_1 join in [11]. The conforming approximation of the new hybrid shell model allows us to reduce the number of degrees of freedom since the new formulation is given in terms of displacement and covariant components of the rotation of normal. Therefore, we introduce a new functional framework for the Naghdi model for shells with G_1-midsurface.

This article is organized as follows. We first recall the geometry of the midsurface and some definitions about G_1 regularity needed to introduce the variational formulation of Naghdi's shell with G_1-midsurface given in [1]. In section 3, we introduce a constraint-free formulation of Naghdi's shell with of G_1-mid surface instead of the one introduced by [1]. We then establish the existence and uniqueness of the solution for a such new model. Section 4 is devoted to the finite element discretization of the hybrid formulation. Let us note that thanks to this constraint-free formulation we only need five degrees of freedom by triangle for computing the solution. Finally, in section 5, we present some numerical tests using FreeFem++ software. We present results of validation for a hyperbolic paraboloid shell and a clamped cylindrical shell. We also show results for two other tests of a basket-handle $W^{2,\infty}$-shell and a hat $W^{2,\infty}$-shell.

2. Notation

Greek indices and exponents take their values in the set $\{1, 2\}$ and Latin indices and exponents take their values in the set $\{1, 2, 3\}$. Unless otherwise specified, the summation convention for indices and exponents is assumed.

Let (e_1, e_2, e_3) be the Cartesian basis of the Euclidean space $\mathbb{R}^3$. We note $u \cdot v$ the inner product of $\mathbb{R}^3$, $|u| = \sqrt{u \cdot v}$ the associate Euclidean norm and $u \wedge v$ the vector product of u and v.

We recall below some definitions and notations introduced in [1].

Definition 1. Let ω be a Lipschitz domain in $\mathbb{R}^2$ and consider a surface chart $\varphi : \overline{\omega} \to \mathbb{R}^3$. We assume that :

1) φ is bilipschitz chart, i.e., there exist two constants $0 < C \leq C'$ such that

$$\forall x, y \in \overline{\omega}, \quad C||y - x|| \leqslant ||\varphi(y) - \varphi(x)|| \leqslant C'||y - x||. \tag{1}$$

2) The unit normal vector is defined almost everywhere by

$$a_3(x) = \frac{\partial_1\varphi(x) \wedge \partial_2\varphi(x)}{||\partial_1\varphi(x) \wedge \partial_2\varphi(x)||} \in W^{1,\infty}(\omega; \mathbb{R}^3). \tag{2}$$

At this point, it will be convenient to introduce a notational shortcut and recall some definitions of [1].

We will say that a surface patch is K-regular if it satisfies (1) and (2).

Definition 2. Let $\omega_1 =]-1, 0[^2$ and $\omega_2 =]0, 1[^2$. And let S_1 and S_2 be two C^2-patches such that $\varphi_1(0, x_2) = \varphi_2(0, x_2)$. Then, S_1 and S_2 have a locally G_1 join around a point $(0, \overline{x})$, $0 < \overline{x} < 1$, if and only if there exists a neighborhood $V_{\overline{x}}$ of $\overline{x}$ such that :

1) Both surfaces have the same tangent plane T_x at point $\varphi_1(0, x)$ for all $x \in V_{\overline{x}}$.

2) For all unit vectors $u = (u_1, u_2)^T$ in $\mathbb{R}^2$, the angle between the vectors $D\varphi_1(0, x)u$ and $-D\varphi_2(0, x)u$ is nonzero.

Let ω be a domain of $\mathbb{R}^2$ such that $\overline{\omega} = \bigcup\limits_{i=1}^{k} \overline{\omega_i}$. We consider a shell whose G_1-midsurface is given by $S = \bigcup\limits_{i=1}^{k} S_i = \bigcup\limits_{i=1}^{k} \varphi_i(\overline{\omega_i})$ where S_i a subdivision of S into patches and $\varphi_i \in W^{2,\infty}(\omega_i; \mathbb{R}^3)$, K-regular is one-to-one mapping such that the vectors

$$a_{\alpha,i} = \partial_\alpha \varphi_i, \quad i = 1, ..., k,$$

are linearly independent at each point $x \in \overline{\omega_i}$. We let

$$a_{3,i}(x) = \frac{a_{1,i}(x) \wedge a_{2,i}(x)}{||a_{1,i}(x) \wedge a_{2,i}(x)||}, \quad i = 1, ..., k,$$

be the unit normal vector on the midsurface at point $\varphi_i(x)$. The vectors $a_{i,j}$ define the covariant basis at point $\varphi_j(x)$. The regularity of the G_1-midsurface chart and hypothesis of linear independence on ω_j imply that $a_{i,j}$ belong to $W^{1,\infty}(\omega_j)$. The contravariant basis a_k^i is defined by the relation $a_{i,k} \cdot a_k^j = \delta_{i,k}^j$, where $\delta_{i,k}^k$ is the Kronecker symbol. In particular $a_{3,i}(x) = a_i^3(x)$. Note that all these vectors are of class $W^{1,\infty}$. We let $a_i(x) = |a_{1,i}(x) \wedge a_{2,i}(x)|^2$ so that $a_i(x)$ is the area element of the G_1-midsurface in the chart φ_i.

The first and second fundamental forms of the surface are given in covariant components by

$$a_{\alpha\beta,i}(x) = a_{\alpha,i}(x) \cdot a_{\beta,i}(x),$$

and

$$b_{\alpha\beta,i}(x) = a_{3,i}(x) \cdot \partial_\beta a_{\alpha,i}(x) = -a_{\alpha,i}(x) \cdot \partial_\beta a_{3,i}(x), \quad \text{since } a_{\alpha,i}(x) \cdot a_{3,i}(x) = 0.$$

Since $W^{1,\infty}$ is a Banach algebra, it follows that $a_{\alpha\beta,i} \in W^{1,\infty}(\omega_i)$ and $b_{\alpha\beta,i} \in L^\infty(\omega_i)$.

Finally, the Christoffel symbols are given by $\Gamma^\rho_{\alpha\beta,i} = \Gamma^\rho_{\beta\alpha,i} = a_i^\rho \cdot \partial_\beta a_{\alpha,i}$, and we have $\Gamma^\rho_{\alpha\beta,i} \in L^\infty(\omega_i)$.

Definition 3. We also assume that ω_i and ω_j are contiguous, i.e., $\omega_i \cap \omega_j = \emptyset$ and $\partial\omega_i \cap \partial\omega_j$ is a segment, we let δ_{ij} denote the common segment on their boundaries.

Similarly, given two functions defined on ω_i and ω_j, which are equal on δ_{ij} if their respective traces on the boundaries agree modulo the translation and rotation that make $\overline{\omega_i}$ and $\overline{\omega_j}$ contiguous.

A regular midsurface displacement will now consist of k mappings $u_i = u_{|\omega_i} \in H^1(\omega_i;\mathbb{R}^3)$ that are equal on δ_{ij} and the same for their regular rotation of the normal vector $r_i = r_{|\omega_i} \in H^1(\omega_i;\mathbb{R}^3)$, i.e., H^1-regular mapping from ω_i into $\mathbb{R}^3$ given in covariant and Cartesian components by :

$$u_j(x) = u_{i,j}(x)a^i_j(x) = u^c_{i,j}(x)e_{i,j} \quad \text{where } u_{i,j} = u_j \cdot a_{i,j} \quad \text{and} \quad u^c_{i,j} = u_j \cdot e_{i,j},$$

and

$$r_j = r_{\alpha,j}(x)a^\alpha_j(x) = r^c_{i,j}(x)e_{i,j} \quad \text{with the same meaning.}$$

Note that the tangency requirement is easily expressed in covariant coordinates, as it simply reads $r_{3,j} = 0$, whereas it becomes

$$r^c_{i,j}(x)a^c_{3,j}(x) = 0 \quad \text{in } \omega_j$$

in Cartesian coordinates.

Let $a_i^{\alpha\beta\rho\sigma} \in L^\infty(\omega_i)$ be the elasticity tensor, which we assume to satisfy the usual symmetries and to be uniformly strictly positive, i.e., for all symmetric tensor $\tau_{\alpha\beta,i}$ and almost all $x \in \omega_i$, we have

$$a_i^{\alpha\beta\rho\sigma}(x)\tau_{\alpha\beta,i}\tau_{\rho\sigma,i} \geqslant c\sum_{\alpha\beta}|\tau_{\alpha\beta,i}|^2 \tag{3}$$

with $c > 0$. To be more specific, we will concentrate on the case of homogeneous, isotropic material with Lamé modulus $\mu > 0$ and $\lambda \geqslant 0$, in which case

$$a_i^{\alpha\beta\rho\sigma} = 2\mu(a_i^{\alpha\beta}a_i^{\rho\sigma} + a_i^{\alpha\sigma}a_i^{\beta\sigma}) + \frac{4\lambda\mu}{\lambda+2\mu}a_i^{\alpha\beta}a_i^{\rho\sigma},$$

where $a_i^{\alpha\beta} = a_i^\alpha \cdot a_i^\beta$ are the contravariant components of the first fundamental form.

Let $e \in L^\infty$ be the thikness of the shell, which we assume to be such that $e(x) \geqslant c > 0$ almost everywhere in ω.

In this context, the covariant components of the change of metric tensor read

$$\gamma_{i,\alpha\beta}(u_i) = \frac{1}{2}(\partial_\alpha u_i \cdot a_{\beta,i} + \partial_\beta u_i \cdot a_{\alpha,i}), \tag{4}$$

the contravariant components of the change of transverse shear tensor read

$$\delta_{i,\alpha 3}(u_i, r_i) = \frac{1}{2}(\partial_\alpha u_i \cdot a_{3,i} + r_i \cdot a_{\alpha,i}), \tag{5}$$

and the covariant components of the change of curvature tensor read

$$\chi_{i,\alpha\beta}(u_i, r_i) = \frac{1}{2}(\partial_\alpha u_i \cdot \partial_\beta a_{3,i} + \partial_\beta u_i \cdot \partial_\alpha a_{3,i} + \partial_\alpha r_i \cdot a_{\beta,i} + \partial_\beta r_i \cdot a_{\alpha,i}), \tag{6}$$

see[1]. Note that all these quantities make sense for shells with little regularity, and are easily expressed with Cartesian coordinates of the unknowns and geometrical data. For instance, we have

$$\partial_\alpha u_j \cdot a_{\beta,j} = \partial_\alpha u^c_{i,j} a^c_{\beta,i,j} \quad \text{and so on.}$$

We assume that the boundary $\partial\omega \cap \partial\omega_i$ of the chart domain is divided into two parts : γ_i^c of strictly positive 1-dimensional measure on which the shell is clamped on a part of its edge and a complementary part γ_i^f on which the shell is subjected to applied tractions and moments.

Let us consider the function space, introduced in [1], which is appropriate in the context of shells with little regularity, i.e, with G_1 midsurface :

$$W = \{(u_i, r_i) \in H^1(\omega_i;\mathbb{R}^3)^2,\ r_i \cdot a_{3,i} = 0 \text{ in } \omega_i,\ u_i = u_j \text{ and } r_i = r_j \text{ on } \delta_{ij}\}. \quad (7)$$

This space is endowed with the natural Hilbert norm

$$\|u\|_W = \Big(\sum_{i=1}^{k} \big(\|u_i\|^2_{H^1(\omega_i;\mathbb{R}^3)} + \|r_i\|^2_{H^1(\omega_i;\mathbb{R}^3)} \big) \Big)^{\frac{1}{2}}. \quad (8)$$

The boundary conditions considered are hard clamping conditions on part of the boundary

$$v_i = 0 \quad \text{and} \quad s_i = 0 \quad \text{on } \gamma_i^c.$$

Let us now recall the formulation of Naghdi's problem with G_1-midsurface. It consists in finding $(u_i, r_i) \in W$ such that

$$\forall (v_i, s_i) \in W, \quad B\big((u_i, r_i), (v_i, s_i)\big) = L\big((v_i, s_i)\big), \quad (9)$$

where

$$\begin{aligned} B\big((u_i, r_i), (v_i, s_i)\big) &= \sum_{i=1}^{k} \int_{\omega_i} \Big\{ e\, a_i^{\alpha\beta\rho\sigma} [\gamma_{i,\alpha\beta}(u_i)\gamma_{i,\rho\sigma}(v_i) + \frac{e^2}{12}\chi_{i,\alpha\beta}(u_i, r_i)\chi_{i,\rho\sigma}(v_i, s_i)] \\ &\quad +4\,\mu\, e\, a_i^{\alpha\beta}\delta_{i,\alpha 3}(u_i, r_i)\delta_{i,\beta 3}(v_i, s_i) \Big\} \sqrt{a_i}\, dx \end{aligned} \quad (10)$$

and

$$L\big((v_i, s_i)\big) = \sum_{i=1}^{k} \int_{\omega_i} P_i \cdot v_i \sqrt{a_i}\, dx + \int_{\gamma_i^f} N_i \cdot v_i + M_i s_i d\gamma. \quad (11)$$

Theorem 1. Let $P_i \in L^2(\omega_i;\mathbb{R}^3)$ be a given resultant force density, $N_i \in L^2(\gamma_i^f;\mathbb{R}^3)$ an applied traction density, $M_i \in L^2(\gamma_i^f;\mathbb{R}^3)$ an applied moment density such that $M_i \cdot a_{3,i} = 0$ almost everywhere on γ_i^f and $e > 0$ the thickness of the shell. Then there exists a unique solution to the problem (9).

Proof.– See [1].

3. An hybrid version of Naghdi's model with G_1-midsurface

The purpose of the present section is to consider our constraint-free formulation for Naghdi model for shells with G_1-midsurface. The unknowns still are the displacement u and rotation r, but instead of considering r_i as a vector field as in [1], we define it by its covariant or contravariant components as usual in shell theory. For more details one can see Ciarlet ([2]).

We firstly define the tensors (4), (5) and (6) in a this new framework. It is the object of the following lemma. Let us recall that the covariant derivatives of the tangential components of $r_{\alpha,i}$ are defined by

$$r_{\alpha,i|\beta} = \partial_\beta r_{\alpha,i} - \Gamma^{\rho}_{\alpha\beta,i} r_{\rho,i}.$$

Lemma 4. Let $u_i \in H^1(\omega_i;\mathbb{R}^3)$, $r_{\alpha,i} \in H^1(\omega_i)$ and $\varphi_i \in W^{2,\infty}(\omega_i)$. Then the covariant components of the change of metric tensor read

$$\gamma_{\alpha\beta,i}(u_i) = \frac{1}{2}(\partial_\alpha u_i \cdot a_{\beta,i} + \partial_\beta u_i \cdot a_{\alpha,i}), \tag{12}$$

the covariant components of the change of transverse shear tensor read

$$\delta_{\alpha 3,i}(u_i, r_{\alpha,i}) = \frac{1}{2}(\partial_\alpha u_i \cdot a_{3,i} + r_{\alpha,i}) \tag{13}$$

and the covariant components of the change of curvature tensor read

$$\chi_{\alpha\beta,i}(u_i, r_{\alpha,i}) = \frac{1}{2}(\partial_\alpha u_i \cdot \partial_\beta a_{3,i} + \partial_\beta u_i \cdot \partial_\alpha a_{3,i}) + \frac{1}{2}(r_{\alpha|\beta,i} + r_{\beta|\alpha,i}). \tag{14}$$

These tensors define functions of $L^2(\omega_i)$.

This leads to the following function space :

$$M = \{(u_i, r_{\alpha,i}) \in H^1(\omega_i;\mathbb{R}^3) \times H^1(\omega_i)^2, u_i = u_j \text{ and } r_{\alpha,i} = r_{\alpha,j} \text{ on } \delta_{ij}\}. \tag{15}$$

Taking into account the boundary conditions, we thus introduce the Hilbert space

$$V = \{(v_i, s_{\alpha,i}) \in M, v_i = s_{\alpha,i} = 0 \quad \text{on} \quad \gamma_i^c\}, \tag{16}$$

and we equipe it with the following norm

$$\|(u, r_\alpha)\|_V = \Big(\sum_{i=1}^{k} \|u_i\|^2_{H^1(\omega_i;\mathbb{R}^3)} + \sum_{\alpha=1}^{2}\sum_{i=1}^{k} \|r_{\alpha,i}\|^2_{H^1(\omega_i)}\Big)^{\frac{1}{2}}. \tag{17}$$

Theorem 2. Let $P_i \in L^2(\omega_i;\mathbb{R}^3)$ be a given resultant force density and $e > 0$ the thickness of the shell. Then there exists a unique solution to the following problem : Find $(u_i, r_{\alpha,i}) \in V$ such that

$$\forall (v_i, s_{\alpha,i}) \in V, \quad a\big((u_i, r_{\alpha,i}), (v_i, s_{\alpha,i})\big) = l\big((v_i, s_{\alpha,i})\big), \tag{18}$$

where

$$\begin{aligned} a\big((u_i, r_{\alpha,i}), (v_i, s_{\alpha,i})\big) \quad &= \sum_{i=1}^{k} \int_{\omega_i} \{e\, a_i^{\alpha\beta\rho\sigma}[\gamma_{\alpha\beta,i}(u_i)\gamma_{\rho\sigma,i}(v_i) + \frac{e^2}{12}\chi_{\alpha\beta,i}(u_i, r_{\alpha,i})\chi_{\rho\sigma,i}(v_i, s_{\alpha,i})] \\ &+4\,\mu\, e\, a_i^{\alpha\beta}\delta_{\alpha 3,i}(u_i, r_{\alpha,i})\delta_{\beta 3,i}(v_i, s_{\alpha,i})\}\sqrt{a_i}\, dx \end{aligned} \tag{19}$$

and

$$l\big((v_i, s_{\alpha,i})\big) = \sum_{i=1}^{k} \int_{\omega_i} P_i \cdot v_i \sqrt{a_i}\, dx. \tag{20}$$

The proof is based on the following version of the infinitesimal rigid displacement lemma.

Lemma 5. Let $u_i \in H^1(\omega_i;\mathbb{R}^3)$, $r_{\alpha,i} \in (H^1(\omega_i))^2$ and $\varphi_i \in W^{2,\infty}(\omega_i;\mathbb{R}^3)$.

(i) If $\gamma_{\alpha\beta,i}(u_i) = 0$, then there exists a unique $\psi \in L^2(\omega;\mathbb{R}^3)$ such that :

$$\partial_\alpha u_i = \psi \wedge \partial_\alpha \varphi_i.$$

(ii) If $\delta_{\alpha 3,i}(u_i, r_{\alpha,i}) = 0$, then $\partial_\alpha u_i \cdot a_{3,i} = -r_{\alpha,i}$ belong to $H^1(\omega_i)$. Furthermore $r_{\alpha,i} = -\varepsilon_{\alpha\beta,i}\psi \cdot a_i^{\beta}$, where $\varepsilon_{11,i} = \varepsilon_{22,i} = 0$ et $\varepsilon_{12,i} = -\varepsilon_{21,i} = \sqrt{a_i}$.

(iii) If, in addition to (i) and (ii), $\chi_{\alpha\beta,i}(u_i, r_{\alpha,i}) = 0$, then ψ is a constant vector in $\mathbb{R}^3$ and there exists $c \in \mathbb{R}^3$ such that

$$u_i(x) = c + \psi \wedge \varphi_i(x).$$

Proof.– We notice that if $\delta_{\alpha 3,i}(u_i, r_{\alpha,i}) = 0$, then $\partial_{\alpha\beta} u_i \cdot a_{3,i} \in L^2(\omega_i)$. Indeed $\partial_{\alpha\beta} u_i \cdot a_{3,i} = \partial_\beta(\partial_\alpha u_i) \cdot a_{3,i} - \partial_\alpha u_i \cdot \partial_\beta a_{3,i} \in L^2(\omega_i)$ as $a_{3,i} \in L^\infty(\omega_i;\mathbb{R}^3)$ and $\partial_\alpha u_i \cdot a_{3,i} = -r_{\alpha,i} \in H^1(\omega_i)$.
Consequently, $\chi_{\alpha\beta,i}(u_i, r_{\alpha,i}) = -(\partial_{\alpha\beta} u_i - \Gamma^{\rho}_{\alpha\beta,i}\partial_\rho u_i) \cdot a_{3,i} = 0$. The results follow the theorem of [11]. For conclusion, just use the infinitesimal rigid displacement lemma for shells with minimal regularity, see [13] and [14].

Lemma 6. There exists a constant $C > 0$ such that

$$a\big((v_i, s_{\alpha,i}), (v_i, s_{\alpha,i})\big) \geqslant \quad C \sum_{i=1}^{k} \Big(\sum_{\alpha,\beta} ||\gamma_{\alpha\beta,i}(v_i)||^2_{L^2(\omega_i)} + \sum_{\alpha,\beta} ||\chi_{\alpha\beta,i}(v_i, s_{\alpha,i})||^2_{L^2(\omega_i)} + \sum_{\alpha} ||\delta_{\alpha 3,i}(v_i, s_{\alpha,i}))||^2_{L^2(\omega_i)}\Big)^{\frac{1}{2}}$$

for all $(v_i, s_{\alpha,i}) \in H^1(\omega_i;\mathbb{R}^3) \times (H^1(\omega_i))^2$.

Proof.– Thanks to inequality (3) and the fact that $a_i^{\alpha\beta}(x)\eta_{\alpha,i}\eta_{\beta,i} \geqslant C' \sum_{\alpha} (\eta_{\alpha,i})^2$ for all $x \in \overline{\omega_i}$.

Lemma 7. We assume that the patches are in $W^{2,\infty}$. The bilinear form in (18) is V-elliptic.

Proof.– Because of lemma (6) and the hypotheses made on the chart φ_i, the elasticity tensor and the thickness of the shell, it is enough to prove that

$$|||(v, s_\alpha)||| = \sum_{i=1}^{k} \Big(\sum_{\alpha\beta} \|\gamma_{\alpha\beta,i}(v_i)\|^2_{L^2(\omega_i)} + \sum_{\alpha\beta} \|\chi_{\alpha\beta,i}(v_i, s_{\alpha,i})\|^2_{L^2(\omega_i)} + \sum_{\alpha} \|\delta_{\alpha 3,i}(v_i, s_{\alpha,i})\|^2_{L^2(\omega_i)}\Big)^{\frac{1}{2}} \tag{21}$$

is a norm on V bounded from below by a multiple of the natural norm (17) of V. Let us first prove that $|||.|||$ is a norm. Let $(v, s_\alpha) \in V$ be such that $|||(v, s_\alpha)||| = 0$. Then, it follows from the infinitesimal rigid displacement lemma 5 that there exists ψ, $c \in \mathbb{R}^3$ such that $v_i(x) = \psi \wedge \varphi_i(x) + c$. Then, we should consider two cases :

1) The displacement v_i vanishes on γ_i^c. If $\varphi_i(\gamma_i^c)$ is not include in a straight line, it follows that $v_i = 0$ a.e. in ω_i, that is to say $\psi = c = 0$. Consequently, we have $s_{\alpha,i} = 0$ a.e. in ω_i as well.

2) Let us now suppose that $\varphi_i(\gamma_i^c)$ is included in a straight line D and that $\psi \neq 0$. In this case, ψ is parallel to D hence belongs to the planes spanned by $(\partial_\alpha \varphi_i)_{|\gamma_i^c}$ for all $x \in \gamma_i^c$. Let us pick one such x. Since $s_{\alpha,i} = 0$ on γ_i^c and it follows that $\partial_\alpha v_i = \psi \wedge \partial_\alpha \varphi_i$, consequently

$$\partial_\alpha v_i \cdot a_{3,i} = \partial_\alpha \varphi_i \wedge a_{3,i} \cdot \psi = 0, \quad \text{on } \gamma_i^c.$$

and ψ is orthogonal to the plane spanned by $(\partial_\alpha \varphi_i)_{|\gamma_i^c}$, which contradicts the hypothesis. Therefore $\psi = 0$ and, as before, $v_i = s_{\alpha,i} = 0$ a.e. in ω_i.
For conclusion we take the remark of [1] : if $\overline{\omega}_i$ and $\overline{\omega}_j$ are two contiguous patches, the equality of traces on δ_{ij} implies that the patched displacement

$$\begin{aligned} \widetilde{u}: \quad \omega \quad &\longrightarrow \mathbb{R}^3 \\ x \quad &\longmapsto \begin{cases} u_i(x) & \text{a.e. if } x \in \omega_i \\ u_j(x) & \text{a.e. if } x \in \omega_j \end{cases} \end{aligned} \tag{22}$$

and patched rotation

$$\begin{aligned} \widetilde{r}_\alpha: \quad \omega \quad &\longrightarrow \mathbb{R} \\ x \quad &\longmapsto \begin{cases} r_{\alpha,i}(x) & \text{a.e. if } x \in \omega_i \\ r_{\alpha,j}(x) & \text{a.e. if } x \in \omega_j \end{cases} \end{aligned} \tag{23}$$

are both in $H^1(\omega;\mathbb{R}^3) \times (H^1(\omega))^2$ where ω is the interior of $\overline{\omega_i} \cup \overline{\omega_j}$. By applying the demonstration made above on $\widetilde{u}$ and $\widetilde{r}_\alpha$ and taking into account the boundary conditions on γ^c (corresponds to portions γ_i^c of $\partial\omega_i$). We obtain

$$\begin{aligned} u_i = 0, \quad \forall i \in \{1,2,...k\}, \quad \text{then} \quad \widetilde{u} = 0 \quad \text{a.e. in } \omega \\ r_{\alpha,i} = 0, \quad \forall i \in \{1,2,...k\}, \quad \text{then} \quad \widetilde{r}_\alpha = 0 \quad \text{a.e. in } \omega \end{aligned}$$

For the second part of the proof, we argue by contradiction. Let us assume that there exists a sequence $(v_n, s_{\alpha,n}) \in V$ such that

$$\|(v_n, s_{\alpha,n})\|_V = 1 \quad \text{but} \quad |||(v_n, s_{\alpha,n})||| \to 0 \quad \text{when} \quad n \to +\infty. \tag{24}$$

By extracting a subsequence, we may assume that there exists $(v, s_\alpha) \in V$ such that

$$(v_{i,n}, (s_{\alpha,i})_n) \rightharpoonup (v_i, s_{\alpha,i}) \quad \text{weakly in } H^1(\omega_i;\mathbb{R}^3) \times (H^1(\omega_i))^2$$

and

$$\gamma_{\alpha\beta,i}(v_{i,n}) \rightharpoonup \gamma_{\alpha\beta,i}(v_i),$$

$$\chi_{\alpha\beta,i}(v_{i,n}, (s_{\alpha,i})_n) \rightharpoonup \chi_{\alpha\beta,i}(v_i, s_{\alpha,i}) \text{ and } \delta_{\alpha 3,i}(v_{i,n}, (s_{\alpha,i})_n) \rightharpoonup \delta_{\alpha 3,i}(v_i, s_{\alpha,i}),$$

weakly in $L^2(\omega_i)$. By hypothesis (24), the three tensors tend strongly to zero in $L^2(\omega_i)$, and using lemma 5 and the discussion above, we infer that $v_i = s_{\alpha,i} = 0$. Then, Rellich's lemma implies that $v_{i,n}$ and $(s_{\alpha,i})_n$ both tend to zero strongly in L^2.

Let us introduce the two-dimensional vector $(w_{i,n})_\alpha = v_{i,n} \cdot a_{\alpha,i}$. We have, $w_{i,n} \to 0$ in $L^2(\omega_i;\mathbb{R}^2)$ strongly. Let us define $2e_{\alpha\beta}(w) = \partial_\alpha w_\beta + \partial_\beta w_\alpha$. It is easy to see that

$$e_{\alpha\beta}(w_{i,n}) = \gamma_{\alpha\beta,i}(v_{i,n}) + v_{i,n} \cdot \partial_\alpha a_{\beta,i} \to 0 \quad \text{strongly in } L^2(\omega_i). \tag{25}$$

Indeed, $\partial_\alpha a_{\beta,i} \in L^\infty(\omega_i)$. Then, by the two-dimensional Korn inequality, we deduce that

$$w_{i,n} \to 0 \quad \text{strongly in } H^1(\omega_i;\mathbb{R}^2). \tag{26}$$

Next we note that

$$\partial_\rho v_{i,n} \cdot a_{\alpha,i} = \partial_\rho\big((w_{i,n})_\alpha\big) - v_{i,n} \cdot \partial_\rho a_{\alpha,i} \to 0 \quad \text{strongly in } L^2(\omega_i). \tag{27}$$

Indeed, $\partial_\rho a_{\alpha,i} \in L^\infty(\omega_i)$. Moreover, as $(s_{\alpha,i})_n \to 0$ in $L^2(\omega_i)$ strongly, and $\partial_\rho v_{i,n} \cdot a_{3,i} = \delta_{\rho 3,i}(v_{i,n}, s_{i,n}) - (s_{\alpha,i})_n$, we already know that

$$\partial_\rho v_{i,n} \cdot a_{3,i} \to 0 \quad \text{strongly in } L^2(\omega_i). \tag{28}$$

We deduce that

$$\partial_\rho v_{i,n} \to 0 \quad \text{in } L^2(\omega_i;\mathbb{R}^3), \tag{29}$$

by (27) and (28). It follows that $v_{i,n} \to 0$ strongly in $H^1(\omega_i;\mathbb{R}^3)$.

We use a similar argument to prove that $(s_{\alpha,i})_n \to 0$ strongly in $H^1(\omega_i)$.

Combining now the convergence of $v_{i,n}$ and $(s_{\alpha,i})_n$, we see that $\|(v_{i,n},(s_{\alpha,i})_n)\|_V \to 0$, which contradicts the hypothesis and proves the lemma.

Lemma 8. For all $i \in \{1,2,\ldots,k\}$, $u_i \in H^1(\omega_i;\mathbb{R}^3)$ then $u \in H^1(\omega;\mathbb{R}^3)$.
For all $i \in \{1,2,\ldots,k\}$, $r_{\alpha i} \in H^1(\omega_i)$ then $r_\alpha \in H^1(\omega)$.

Proof.– If $u_i \in H^1(\omega_i;\mathbb{R}^3)$ such that $u_i = 0$ on γ_i^c and $u_i = u_j$ on δ_{ij} then $u \in H^1(\omega;\mathbb{R}^3)$ and $u = 0$ on γ^c. If $r_{\alpha i} \in (H^1(\omega_i))^2$ such that $r_{\alpha i} = 0$ on γ_i^c and $r_{\alpha i} = r_{\alpha j}$ on δ_{ij} then $r_\alpha \in (H^1(\omega_i))^2$ and $r_\alpha = 0$ on γ^c.

Proof.– (of theorem 2)
The bilinear and linear forms of problem (18) are clearly continuous on the space V. We have just shown that the bilinear form is V-elliptic. We use the Lax-Milgram theorem to conclude.

4. Finite Element Approximation

4.1. A finite element formulation

Let us first introduce some notation. Let(T_{h_i}) be a shape-regular family of triangulations of ω_i. For a triangle K_i in T_{h_i}, let h_{K_i} be its diameter and set $h_i = \max_{K_i} h_{K_i}$. We denote by S_{h_i} the set of vertices in T_{h_i}. For any subdomain Δ_i of ω_i and for $k \geq 0$ we define by $\mathbb{P}_{k_i}(\Delta_i)$ the space of polynomials on Δ_i with degree $\leq k$. In what follows, C denotes a constant independent of h_i.

The discrete space of admissible displacements and rotations is given by

$$V_h = \{(v_{h,i}, s_{\alpha,h,i}) \in M_h, v_{h,i} = s_{\alpha,h,i} = 0 \quad \text{on } \gamma_i^c\} \tag{30}$$

where

$$M_h = \{(v_{h,i}, s_{\alpha,h,i}) \in C^0(\omega_i;\mathbb{R}^3)\times C^0(\omega_i)^2, \forall K_i \in T_{h_i}\ (v_{h,i}, s_{\alpha,h,i})_{|K_i} \in \mathbb{P}_1(K_i;\mathbb{R}^3)\times\mathbb{P}_1(K_i)^2,$$
$$v_{h,i} = v_{h,j}\ \mathit{and}\ s_{\alpha,h,i} = s_{\alpha,h,j}\ \mathit{on}\ \delta_{ij}\}.$$

Thus the discrete problem reads : Find $(u_{h,i}, (r_{\alpha,h,i})) \in V_h$ such that

$$\forall (v_{h,i}, s_{\alpha,h,i}) \in V_h, \ \ a((u_{h,i}, r_{\alpha,h,i}), (v_{h,i}, s_{\alpha,h,i})) = l(v_{h,i}, s_{\alpha,h,i}) \tag{31}$$

Naturally, this problem has a unique solution.

4.2. Convergence

By virtue of the classical properties of Galerkin approximation, we have the following convergence results

Theorem 3. There exists a sequence $h \to 0$ such that

$$\|(u_i, r_{\alpha,i}) - (u_{h,i}, r_{\alpha,h,i})\|_V \to 0 \tag{32}$$

Proof.– We have $u_{h,i} \to u_i$ when $h \to 0$ by virtue of properties of Galerkin approximation of a classical variational problem.

If the solution is assumed to have some regularity, the second step of the approximation may of course be controlled via an error estimate.

Proposition 9. Assume that the solution $(u_i, r_{\alpha,i})$ of problem (18) belongs to $H^2(\omega_i, \mathbb{R}^3) \times (H^2(\omega_i))^2$. Then there exists a constant C independent of h, such that

$$\|(u_i, r_{\alpha,i}) - (u_{h,i}, r_{\alpha,h,i})\|_V \leq C\ h\ \|(u_i, r_{\alpha,i})\|_{H^2(\omega_i,\mathbb{R}^3)\times(H^2(\omega_i))^2} \tag{33}$$

Proof.– See [10], for example.

Remarks 10. i) The *a priori* error estimate given in Proposition 9 makes an essential use of a H^2-regularity of the solution.

ii) For our Hybrid Naghdi's formulation of the present work and also for Cartesian equations where the surface is globally $W^{2,\infty}$, the solution regularity is still an open problem.

iii) The solution regularity depends on the regularity of chart defining the shell midsurface. In the case where the chart is of class C^3, O. Iosifescu in [17] has established the H^2-regularity of the solution in a local covariant or contravariant framework.

iv) In a recent work [16], I. Merabet and S. Nicaise have introduced a new mixed formulation for Nagdhi's shell which is appropriate for folded surfaces and have approximated the solution of the problem using the DK method. The standard a priori error analysis of such methods uses additional regularity on the solution but in [16], I. Merabet and S. Nicaise have carried out an error analysis which only requires the regularity of the weak solution.

5. Numerical experiments

In this section, we implement the discretization of the hybrid approach. We compare it on a literature benchmark for validation. We also apply it to shells with curvature discontinuities where the midsurface is G_1.

5.1. Implementation details

The model formulation only require the knowledge of $a_{\alpha,i}$, $a_{3,i}$, and $\partial_\alpha a_{3,i}$. All other quantities, either geometrical like the elasticity tensor or kinematical like the strain tensors, can be expressed by means of dot products involving these quantities. It is convenient to define these vectors as FreeFem++ functions. The dot products are expressed as FreeFem++ macros, which are then combined into other macros that eventually expand to all the other quantities of interest. The net result is that our code automatically constructs the bilinear forms, with minimal user input. This works well if an analytic description of the G_1-midsurface is available.

We note that with respect to user input and code complexity, our approach compares favorably with classical formulations which require the computation of the covariant and mixed components of the second fundamental form and of the Christoffel symbols of the chart ; see, for example, [9].

Three-dimensional visualization of the undeformed and deformed shells uses Medit, a free mesh visualization software available at `http://www.ann.jussieu.fr/frey/logiciels/medit.html`.

All the tests were run on 4GB HP 15 Notebook laptop.

5.2. Numerical validation

We present in this section two tests. The first is a literature benchmark for shell elements. The second is a cylindrical shell.

5.2.1. The hyperbolic paraboloid shell

The midsurface of the hyperbolic paraboloid shell is represented by a chart of class C^∞.

The reference domain of the midsurface is given by

$$\omega = \{(x, y) \in \mathbb{R}^2, |x| + |y| < \sqrt{2}b\},$$

and the chart is defined by

$$\varphi(x, y) = (x, y, \frac{c}{2b^2}(x^2 - y^2))^T,$$

with $b = 50$ cm and $c = 10$ cm. The thickness of the shell is $e = 0.8$ cm.

The shell is clamped on $\partial\omega$ and subjected to a uniform pressure $q = 0.01$ kp/cm^2. The mechanical data are

$$E = 2.85 \times 10^4 \text{ kp/cm}^2, \quad \nu = 0.4$$

The reference value for this test is the normal displacement at the center $O(0, 0)$ of the shell. Its value computed by various methods is of -0.024 cm ; see [9].

Due to the symmetries of the problem, we use the computational domain

$$\omega' = \{(x, y) \in \mathbb{R}^2, 0 < x, 0 < y, x + y < \sqrt{2}b\},$$

and enforce the symmetry conditions

$$u_2 = 0, \quad r_2 = 0, \quad \text{on } y = 0, \quad \text{and} \quad u_1 = 0, \quad r_1 = 0 \quad \text{on } x = 0.$$

These conditions are obtained by expressing the continuity of the three-dimensional Kirchhoff-Love displacement along these edges. In Table 1. the value of normal displacement at O is reported for hybrid and mixed formulation. Those values are given in respect of the number of degrees of freedom.

Tableau 1 – Hybrid formulation *vs* mixed formulation

Formulation	Dof	$u_3(0,0)$
Hybrid	10615	- 0.0242544
Mixed	14861	- 0.0242545

The numerical results obtained for the reference value are very satisfactory as they are close to -0.024 cm, the value given in the literature. We note that the hybrid method achieves excellent performance in terms of the reference value and with less degrees of freedom, which is in good agreement. For qualitative results we plot the isovalues of u_3 below in Figure (1), solution of the hybrid equations.

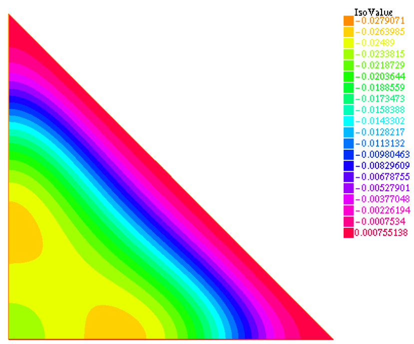

Figure 1 – Qualitative result : Isovalues of u_3.

5.2.2. A clamped cylindrical shell

In this example, we consider a cylindrical shell which is submitted to a uniform pressure load, and clamped only on the top. The opposite side being free. The reference domain of the midsurface is given by

$$\omega = \{(x, y) \in \mathbb{R}^2; 0 < x < 2\pi R, 0 < y < H\},$$

and the chart is defined by

$$\varphi(x, y) = (R\cos(x), R\sin(x), y)^t,$$

where $H = 20$cm and $R = 10$cm. The thickness of the shell is $e = 0.5$ cm. The shell is clamped on $\gamma^e = \{(x, H) \in \mathbb{R}^2; 0 < x < 2\pi R\}$ and subjected to an internal pressure $p = 1$kp/cm^2. The mechanical data are $E = 3 \times 10^5$kp$/cm^2$, and $\nu = 0.22$. The analytic normal displacement solution of the problem is given by :

$$u_3 = K(1 + \exp(\mu y)(A\cos(\gamma y) + B\sin(\gamma y)) + \exp(-\mu y)(C\cos(\gamma y) + D\sin(\gamma y)))$$

where

$$K = \frac{3p(1-\nu^2)}{2e^3 E}\frac{e^2R^4}{e^2 + 3R^2(1-\nu^2)}, \quad \mu = \frac{\sqrt{c^2 - b}}{\sqrt{2}}, \quad \gamma = \frac{\sqrt{c^2 + b}}{\sqrt{2}},$$

$$\text{with} \quad b = \frac{\nu}{R^2}, \quad c^4 = \frac{1}{R^4} + \frac{3(1-\nu^2)}{\varepsilon^2 R^2},$$

and

$$(A, B, C, D) = (-0.0001466, -0.0003600, -0.0058966, 0.0061476).$$

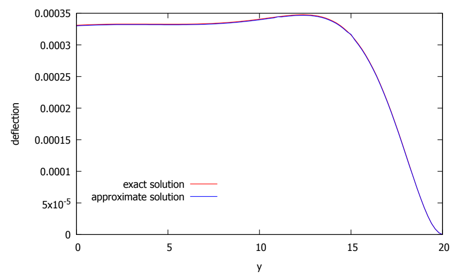

Figure 2 – Exact and approximate solutions of u_3

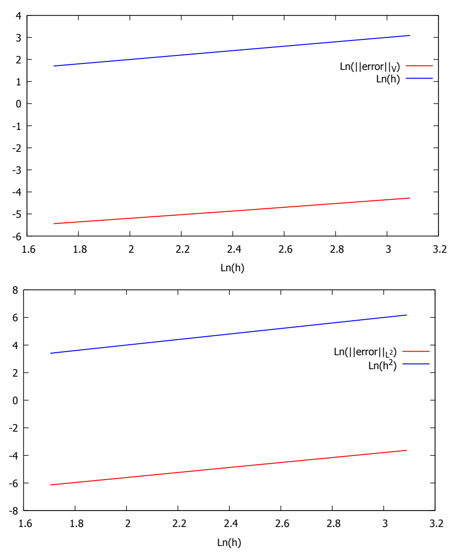

Figure 3 – Convergence rate for the displacement in V norm (up). Convergence rate for the displacement in L^2 norm (down).

We plot in Fig 2. the exact and approximate solutions of the normal covariant displacement of the midsurface. We thus establish the error between the approximate and exacte solutions in V norm as well as in L^2 norm. In Fig 3. we show the convergence rates which confirm the theoretical estimates.

5.3. Other tests

5.3.1. A basket-handle tunnel

Our next test is a genuine $W^{2,\infty}$ test with curvature discontinuities. The shell is made of 3 cylindrical parts with different radiuses and C^1-joins. It is a tunnel-like shell based on a slightly extended 3-circles basket-handle arc.

The mechanical data are given by $E = 3 \times 10^6$ psi, $\nu = 0.0$ and the thickness of the shell e equals 3 in. The shell is submitted to a uniform downward pressure of -0.625 lb/in^2.

The reference domain of the midsurface is given by, $\omega = \bigcup_{i=1}^{3} \omega_i =]-\frac{4\pi}{9}R, \frac{4\pi}{9}R[\times]-L, L[$ where

$$\omega_1 =]-\frac{4\pi}{9}R, -\frac{2\pi}{9}R[\times]-L, L[,$$

$$\omega_2 =]-\frac{2\pi}{9}R, \frac{2\pi}{9}R[\times]-L, L[,$$

and

$$\omega_3 =]\frac{2\pi}{9}R, \frac{4\pi}{9}R[\times]-L, L[,$$

and the chart is defined by

$$\begin{array}{llll} \varphi : & \omega & \longrightarrow \mathbb{R}^3 & \\ & (x,y) & \longmapsto & \left\{ \begin{array}{ll} \left(\frac{R}{3} + \frac{2R}{3}\sin(\frac{3x}{2R} + \frac{\pi}{6}), y, \frac{2R}{3}\cos(\frac{3x}{2R} + \frac{\pi}{6})\right) & \text{if } (x,y) \in \omega_1 \\ \left(\frac{4R}{3}\sin(\frac{3x}{4R}), y, -\frac{R}{\sqrt{3}} + \frac{4R}{3}\cos(\frac{3x}{4R})\right) & \text{if } (x,y) \in \omega_2 \\ \left(\frac{R}{3} + \frac{2R}{3}\sin(\frac{3x}{2R} - \frac{\pi}{6}), y, \frac{2R}{3}\cos(\frac{3x}{2R} - \frac{\pi}{6})\right) & \text{if } (x,y) \in \omega_3 \end{array} \right. \end{array}$$

with $R = 300$ in and $L = 300$ in.

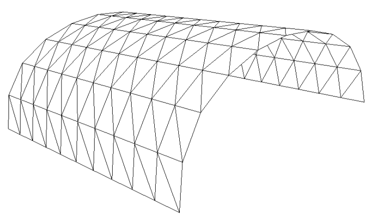

Figure 4 – The mesh on the basket-handle midsurface.

Concerning boundary conditions, we consider the case of hard clamping on both rectilinear sides of the shell. The large circle radius is 400 in. and the small circle radius 200 in. (We compute the whole shell without using the symmetries for better visualization.)

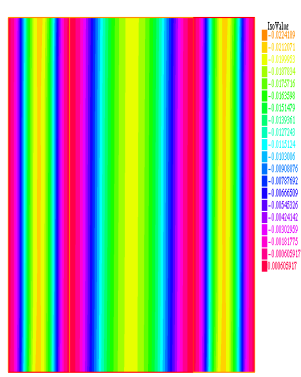

Figure 5 – Qualitative result : Isovalues of u_3.

Now, for the same shell we consider G_1-parametrization instead of the C^1 one. We thus propose below a G_1 chart on the same reference domain :

$$\varphi : \quad \omega \quad \longrightarrow \mathbb{R}^3$$
$$(x,y) \quad \longmapsto \left\{ \begin{array}{ll} \varphi_1(x,y) = \left(\frac{R}{3} + \frac{4R}{3}\sin(\frac{3x}{2R}+\frac{\pi}{6}), y, \frac{4R}{3}\cos(\frac{3x}{2R}+\frac{\pi}{6})\right) & \text{if } (x,y)\in\omega_1 \\ \varphi_2(x,y) = \left(\frac{4R}{3}\sin(\frac{3x}{4R}), y, -\frac{R}{\sqrt{3}} + \frac{4R}{3}\sin(\frac{3x}{4R})\right) & \text{if } (x,y)\in\omega_2 \\ \varphi_3(x,y) = \left(\frac{R}{3} + \frac{4R}{3}\sin(\frac{3x}{2R}-\frac{\pi}{6}), y, \frac{4R}{3}\cos(\frac{3x}{2R}-\frac{\pi}{6})\right) & \text{if } (x,y)\in\omega_3 \end{array}\right.$$

Let us compute the tangent vectors at join points $(-\frac{2\pi}{9},0)$ and $(\frac{2\pi}{9},0)$.

On $x = (-\frac{2\pi}{9},0)$, we get

$$a_{11} = \frac{\partial\varphi_1}{\partial x} = (\sqrt{3},0,1)^t \quad \text{and } a_{12} = \frac{\partial\varphi_2}{\partial x}(\frac{\sqrt{3}}{2},0,\frac{1}{2})^t$$

We thus obtain $a_{11} = 2a_{12}$.

On $x = (\frac{2\pi}{9},0)$, we obtain

$$a_{13} = \frac{\partial\varphi_3}{\partial x} = (\sqrt{3},0,-1)^t \quad \text{and } a_{12} = \frac{\partial\varphi_2}{\partial x} = (\frac{\sqrt{3}}{2},0,-\frac{1}{2})^t$$

and then we find $a_{13} = 2a_{12}$.

The reference value for this test is the normal displacement at the center $O(0,0)$ of the shell. In Table 2. we present the values of normal displacement at O and the number of degrees of freedom which are obtained from mixed method with C^1-parametrization, hybrid method with C^1-parametrization and hybrid method with G_1-parametrization, using the same geometry with $\mathbb{P}_2$ elements.

Tableau 2 – Comparison of resolution methods

Formulation	Dof	$u_3(0,0)$
Mixed with C^1-parametrization	693	- 0.02178
Hybrid with C^1-parametrization	495	- 0.02161
Hybrid with G_1-parametrization	525	- 0.02147

We note that the hybrid method achieves excellent performance in terms of the reference value and with less degrees of freedom, which is in good agreement. Figure (4) presents the initial mesh and for qualitative results we plot the isovalues of u_3 in figure (5), solution of the hybrid equations with G_1-parametrization.

5.3.2. Plate-cylinder

We consider a real test of a $W^{2,\infty}$ shell with curvature discontinuities. Let a shell formed by a flat part, called plate, and a cylindrical portion, which are connected in a C^1-way. See figure (6).

To solve the problem of this shell, we use the hybrid method and we compared it with the mixed method . The reference domain of the midsurface is given by $\omega =]-L,L[\times]-\frac{Ld}{2},\frac{Ld}{2}[$, and the chart is defined by :

$$\varphi(x,y) = \left\{ \begin{array}{ll} (x,y,0)^t & \text{if } x<0, \\ (R\sin(x/R), y, R(1-\cos(x/R)))^t & \text{if } x\geq 0, \end{array}\right. \tag{34}$$

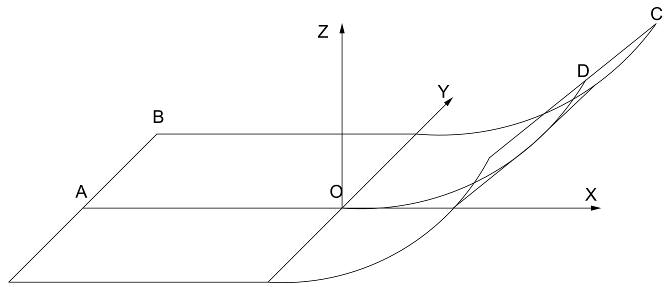

Figure 6 – Shell plate-cylinder.

with $L = 2R\dfrac{\pi}{9}$ in., $R = 300$ in. and $Ld = 600$ in.

The thickness of the shell e equals 7.5 in. and the mechanical data are given by $E = 2.1 \times 10^4\ psi,\ \nu = 0.0$. The shell is submitted to a uniform downward pressure of $P = 0.625$ Lb/in^2.

Concerning boundary conditions, we consider the case of hard clamping on the lines AB and DC

$$u_1 = u_2 = u_3 = 0, \quad \text{and} \quad r_1 = r_2 = 0,$$

and on the remaining edges, it is assumed that the shell is free. Thanks to symmetries, we consider only the half of the midsurface, $y > 0$. The corresponding symmetry conditions on AD are

$$u_2 = r_2 = 0.$$

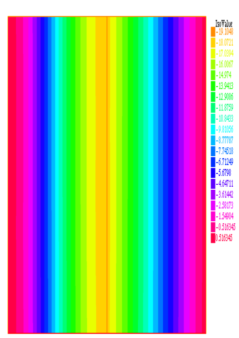

Figure 7 – Qualitative result : Isovalues of u_3.

Now, for the same shell we consider G_1-parametrization instead of the C^1 one. We thus propose below a G_1 chart on the same reference domain :

$$\varphi(x,y) = \begin{cases} (x, y, 0)^t & \text{if } x < 0, \\ (2R\sin(x/R), y, R(1-\cos(x/R)))^t & \text{if } x \geq 0, \end{cases} \tag{35}$$

Let us compute the tangent vectors at join point $(0, y)$.
We obtain

$$a_{11} = \frac{\partial \varphi_1}{\partial x} = (1,0,0)^t \text{ and } a_{12} = \frac{\partial \varphi_2}{\partial x} = (2,0,0)^t.$$

We thus obtain $a_{11} = 2a_{22}$.

The reference value for this test is the normal displacement at the center $O(0,0)$ of the shell. Table 3. present the values of normal displacement at O and the number of degrees of freedom which are obtained from mixed method with C^1-parametrization, hybrid method with C^1-parametrization and hybrid method with G_1-parametrization, using the same geometry with $\mathbb{P}_1$ elements.

Tableau 3 – Comparison of resolution methods

Formulation	Dof	$u_3(0,0)$
Mixed with C^1-parametrization	539	- 18.1381
Hybrid with C^1-parametrization	385	- 18.1384
Hybrid with G_1-parametrization	420	- 18.1323

We note that the hybrid method achieves excellent performance in terms of the reference value and with less degrees of freedom, which is in good agreement. For qualitative results we plot the isovalues of u_3 in figure (7), solution of the hybrid equations with G_1-parametrization.

6. Conclusion

In this paper we have presented a new version of Naghdi's shell with G_1-midsurface. We thus introduce an hybrid Naghdi problem in which the unknowns are the displacement u and the rotation r, respectively elements of the spaces $H^1(\omega;\mathbb{R}^3)$ and $(H^1(\omega))^2$ without any functional constraint. For this hybrid model we have derived a finite element scheme for which we have established existence and uniqueness of the solution as well. Its discretization requires less degrees of freedom than the mixed formulation of [5].

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