Assessing Maintenance of Arboviruses in Nature via *Aedes* Mosquitoes by Positive semigroup

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Abstract
For more than one century, *Aedes* species are supposed to be a reservoir in dengue, yellow fever, rift valley fever and west nile viruses transmission. In this article, we study an infinite dimension ordinary differential equations system that models arbovirus vertical transmission in *Aedes* mosquito. Relying of the positive semigroup theory, we show that the model is well-posed and compute a threshold parameter known as the basic reproduction ratio $\mathcal{R}_0$. This parameter describes "the average rate of secondary new cases of infected adult females from emergences in a breeding habitat that are produced by an infected adult female via transovarial transmission during its lifetime." In addition, we prove that the solution of the model goes to zero asymptotically if $\mathcal{R}_0 < 1$, else it has the property of balanced exponential growth. Finally, a climate-environment effects Index on model parameters and a diagram depicting the conditions of arboviruses persistence via *Aedes* in nature is derived.

Keywords
Positive semigroup, spectral theory, quasi-compactness, balanced exponential growth, basic-ecological reproduction rate number *Aedes* borne-disease, transovarial transmission.

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I INTRODUCTION

In managing the risk of many *Aedes* mosquito borne diseases emergence and resurgence like dengue, yellow fever, rift valley fever, west nile, it has been asked the role of vertical transmission in mosquito species for epidemiological control since transovarial transmission was discovered more than a century ago [2, 11, 13, 17, 22, 25, 27, 28, 30, 32]. Indeed, while known that only adult females assure indirect horizontal transmission between vertebrate hosts from blood meals involved in mosquito ovarian maturation and oviposition [3, 46]. Morever the *Aedes* subspecies population involved in a specific arbovirus transmission can contamine its offsprings via eggs [4, 16, 24, 40] . So, this mode of transmission, called transovarial transmission, provided in nature a system of infected Eggs-Adults for floodwater *Aedes* mosquito species (shortly *Aedes spp.*) which lay eggs on depressions, on damp soils or above mean high water [29, 33, 36, 38, 39]. In such systems, it has been recognized that eggs observe an irreducible period of drying or diapause before hatching during their future submersions of water from rain or another source [12, 20, 34, 41, 42]. But, hatching events are not known to be uniform for a given batch which lifetime span from weeks to few years [10, 31, 37].

In this paper, we study a mathematical model describing a system composed of infected eggs by transovarial transmission of floodwater *Aedes* divided into compartments of individuals that experienced n flooding events and the class of infected adult females. We denote by $u_n(t)$ the density of infected eggs in the system that have experiencing n flooding event(s) and $N(t)$ the density of adult females at time t, respectively. We suppose that those individuals live in a closed *Aedes* breeding habitat free of another reservoir of the arbovirus in interest with favorable climate and environment conditions of adult females life and eggs development from laying to adult juvenile emergences. Besides, we assume that there is not any infected or infectious active host in the area of experimentation during this study and neglect mosquito aquatic stages without loss generality. Then, the model describing transovarial transmission of an arbovirus between adult females and eggs of *Aedes spp.* reads as follows:

$$\begin{cases} \frac{du_0(t)}{dt} &= \beta p N(t) - \alpha_0 u_0(t) - \gamma_0 u_0(t), \\ \frac{du_n(t)}{dt} &= \gamma_{n-1} u_{n-1}(t) - \alpha_n u_n(t) - \kappa_n u_n(t) - \gamma_n u_n(t), n \geq 1, \\ \frac{dN(t)}{dt} &= \sum_{n=1}^{+\infty} \kappa_n u_n(t) - \mu N(t). \end{cases} \tag{1}$$

with initial conditions

$$u_n(0) := u_n^0 \geq 0, \quad \text{for all } n \in \mathbb{Z}_+ := \{0, 1, 2, ...\} \text{ and } N(0) := N_0 \geq 0.$$

Here p and μ denote the female mosquito population probability of transovarial transmission and mortality rate respectively. The expression $\beta p N$ represents the production rate of infected eggs in the system by adults where βp denotes the mean production rate of infected eggs by an adult female and β denotes the egg-laying rate parameter. The model parameters α_n, γ_n and κ_n denote the mortality rate of eggs in n-state of flooding, the transition rate between n and $n+1$-states and the hatching rate of n-state compartment, respectively.

A particular case study of this model (1), based on diapause of mosquito eggs phenomena is not uniformly broken in water of reduced oxygen content during submersion of embryonate eggs which have spent an irreducible drying period after one or successive cycles of flooding-drying [20, 23, 31, 42], has been treated by *Bicout et al* in [10]. To overcome mathematical study difficulties, they exhibited, by approximation, a particular and analytical solution of the model when assumed in one hand that the flooding frequency (γ) and the eggs lifetime (β^{-1}) are constant. In the other hand, the adult mortality rate is supposed to be equal to egg-laying rate parameter ($\beta = \mu$) and the eggs hatching rate function increases linearly per flooding event ($\kappa_n = \kappa n$; κ constant). Thereafter, they derived a parameter playing likely the basic reproduction rate number role from thresholding persistence time and numerical simulations.

Actually, we have not in our disposal a theoretical study of the particular case study of (1). But, several studies through famous deterministic and finite dimension host-vector infection mathematical models examined the epidemiological or enzootic impacts of *Aedes* vertical transmission in various arbovirual diseases outbreaks [1, 14, 15, 35]. Generally, these models described, beyond the classic relations between health status compartments as susceptible-exposed-infectious-recovered pattern for hosts and susceptible-exposed-infectious for vectors, the vertical transmission in *Aedes* mosquito. They commonly consider Van Den Driessche [44] or Diekmann [18, 19, 26] spectral radius of next generation matrix determination approach and gave a threshold parameter coinciding sometime to May and Anderson basic reproduction number definition[5], $\mathcal{R}_0$. They showed that if $\mathcal{R}_0 < 1$, then the equilibrium without disease is locally asymptotically stable and the disease cannot invade host-vector populations; else unstable and the disease possibly persists between them. These studies combined threshold parameters sensitivity analyses and numerical simulations incorporated seasonality or diapause patterns with troublesome host implications in the virus processes maintenance in ecosystems in order to keep climate and environment effects ($e.g$ see [14] for valley fever and [1] for dengue). They suggested various and apparently reversed transovarial transmission epidemiologicalimpacts according to the arbovirus, the *Aedes* subspecies, the hosts and the environmental conditions involved.

The purpose of this work is to highlight the conditions of which *Aedes* mosquito can be a reservoir or not for a specific arbovirus in different ecosystems by using perturbations and spectral theories in positive semigroup approach. In this way, we shall prove that under suitable assumptions on parameters, the model (1) describing only arbovirus transovarial transmission between Mosquito Adult female and eggs is globally well-posed, extract the reproduction rate number ($\mathcal{R}_0$) of the system and provide its solution possesses the properties of asynchronous exponential growth when $\mathcal{R}_0 \geq 1$, in contrary it converges to zero as time tends to infinity. The spectral bound of the differential system (1) operator that matches to the intrinsic growth value of *Aedes* population in its environment [43, 47] is shown to be equivalent to $\mathcal{R}_0$ as bifurcation parameter. Thereafter, we showed that results on asymptotic behavior of the model (1) govern the qualitative analysis which maps, by a diagram, some conditions of arbovirus persistence in nature through only floodwater *Aedes* vertical transmission. In the model under this study, we assume, more general assumptions than *Bicout* and co-workers did in [10], for the death, transition, hatching and egg-laying parameters of n-states in accordance to they vary in nature so that flooding does not systematically hatch mosquito eggs ($e.g$, see [34]). Indeed, throughout this paper, we assume the following hypotheses:

(H1) the eggs death rates are bounded $i.e$ there exist two numbers $\hat{\alpha}$, and $\check{\alpha}$ so that for any integer n ($n \in \mathbb{Z}_+$):
$$0 < \check{\alpha} \leq \alpha_n < \hat{\alpha};$$

(H2) the flooding rates are bounded: there exists a number $\hat{\gamma}$ so that for all $n \in \mathbb{Z}_+$
$$0 < \gamma_n \leq \hat{\gamma};$$

(H3) the eggs hatching rates cannot only increasing and satisfy
$$\forall\, n \in \mathbb{Z}_+, \;\; \kappa_n \geq 0 \;\text{ and }\; \forall\, n_0 \in \mathbb{Z}_+ \quad \exists\, n \geq n_0 \quad \text{so that} \quad \kappa_n > 0;$$

(H4) the adults egg-laying and mortality rate parameters satisfy:
$$\beta > 0 \text{ and } \mu > 0.$$

In addition, we consider the Banach space $E = l^1(\mathbb{Z}_+) \times \mathbb{R}$ with norm $\|(u,N)\|_E = \|u\|_{l^1(\mathbb{Z}_+)} + |N|$ for all $(u,N) \in l^1(\mathbb{Z}_+) \times \mathbb{R}$ where $\mathbb{Z}_+ = \{0,1,2,\cdots,\}$, $l^1(\mathbb{Z}_+) = \{u = (u_n)_{n\in\mathbb{Z}_+} \subset \mathbb{R} \ : \sum_{n=0}^{\infty} |u_n| < \infty\}$ and $\|u\|_{l^1(\mathbb{Z}_+)} = \sum_{n=0}^{\infty} |u_n| \quad \forall \ u \in l^1(\mathbb{Z}_+)$ denote the set of positive integers, the absolutely convergent series of real numbers space and norm, respectively. For keeping biological processes positivity, this space E is ordered by its usual positive cone $l^1(\mathbb{Z}_+)_+ \times \mathbb{R}_+ = \{u = (u_n)_{n\in\mathbb{Z}_+} \in l^1(\mathbb{Z}_+) \ : \forall n \in \mathbb{Z}_+ \quad u_n \geq 0\} \times \mathbb{R}_+$ as $x \geq y$ iff $x - y \in l^1(\mathbb{Z}_+)_+ \times \mathbb{R}_+$ which we denote by $E_+ = \{x \in E;\ x \geq 0\}$ in order to define positive vectors and a positive linear operator L on E as: $0 \leq x \in E$ implies $Lx \geq 0$ as well as $x >> y$ for $x = (x_i)_{i\in Z_+}$ and $y = (y_i)_{i\in Z_+}$ denotes $x_i > y_i,\ \forall i \in \mathbb{Z}_+$.
The standard following AL-space properties satisfied by E are also considered: every pair $x, y \in E$ has both supremum and infimum

$$\forall x \in \ E \ \ x = x_+ - x_- \ \ , \ \ |x|_v = x_+ + x_-, \tag{2}$$

where

$$x_+ = \max\{x,0\},\ x_- = \max\{-x,0\} \text{ and } |x|_v = \max\{-x,x\}; \tag{3}$$

$$\|x\|_E = \||x|_v\|_E \text{ for all } x \in E, \tag{4}$$

$$|x|_v \leq |y|_v \text{ implies } \|x\|_E \leq \|y\|_E \text{ for all } x,\ y \in E \tag{5}$$

and

$$\|x+y\|_E = \|x\|_E + \|y\|_E \text{ for all } x,\ y \in E_+. \tag{6}$$

Also, we denote Our first result considers the well-posedness of the model (1) as an initial value problem which operator, denoted by A, will be defined in section II. It reads as follows.

Theorem I.1:
Let assumptions (H1), (H2), (H3) and (H4) be satisfied. Then, the operator A of (1) is generator of an infinitesimal positive C_0-semigroup, $(T_A(t))_{t\geq 0}$, on E.

The proof of this result will be given in Section II. This Theorem means that for any initial positive conditions $(u_0, N_0) \in l^1(\mathbb{Z}_+)_+ \times \mathbb{R}_+$ at a starting time $t_0 \geq 0$, we have the unique solution $(u(t), N(t))$ of (1) remains positive for $t \geq t_0$, $i.e$,

$$(u(t), N(t)) = T_A(t)(u_0, N_0) \in l^1(\mathbb{Z}_+)_+ \times \mathbb{R}_+ \ \forall\, t \geq t_0.$$

Our second result is devoted to the asymptotic behavior of model's (1) solutions. Before stating this result, we determine from real eigenvalue problem of (1), the formula of the reproduction rate number ($\mathcal{R}_0$) as follows:

$$\mathcal{R}_0 = \frac{\beta p}{\mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{\alpha_n + \kappa_n + \gamma_n} \prod_{i=0}^{n-1} \frac{\gamma_i}{\alpha_i + \kappa_i + \gamma_i}. \tag{7}$$

This $\mathcal{R}_0$ states as a bifurcation parameter of the system (1) in the manner that our main result reads as follows.

Theorem I.2:
The positive C_0-semigroup $(T_A(t))_{t\geq 0}$ is irreducible and the following assertions hold:

i) if $\mathcal{R}_0 < 1$ then: $\exists\, \varepsilon > 0,\ M \geq 1:\ \left\|T_A(t)\right\| \leq Me^{-\varepsilon t}$,

ii) if $\mathcal{R}_0 \geq 1$ then there exist $\lambda_0 \geq 0$, $\varepsilon > 0$, $M \geq 1$ and a projector P such that:
$$PT_A(t) = T_A(t)P = e^{\lambda_0 t}P$$
and
$$\left\|e^{-\lambda_0 t}T_A(t)(I-P)\right\| \leq Me^{-\varepsilon t}$$
where $P(x) = \Phi(x)V,\ \forall x \in E$ with $V \in E_+$ a quasi-interior point and $\Phi \in E^*$ (topological dual of E) satisfying $\Phi(x) > 0,\ \forall x > 0$ and $\phi(V) = 1$.

The proof of this result will be given in Section III.
Thereafter, the $\mathcal{R}_0$ formula and the assertions in the theorem allows to derive, as an illustrative application, both a dimensionless Index, $\mathcal{E}_{C0}$, and a diagram mapping the conditions for which an *Aedes* subspecies can be reservoir of a specific virus or not.

The layout of the rest part is as follows. In Section II, we reduce model (1) into an abstract Cauchy problem and establish its well-posedness by means of strongly continuous positive semigroups. In Section III, we prove that the solution of model (1) goes to zero in the vicinity of infinity if $\mathcal{R}_0 < 1$, else it has asynchronous exponential growth. The conception of a map designing the climatic and environmental effects on mosquito demographic parameters involved in the maintain of arboviruses in nature by *Aedes* is shown in the section IV.

II WELL-POSEDNESS OF THE MODEL

In this section, we use the positive semigroup theory to show the well-posedness of the model (1). Indeed, we shall recall some characteristics of the spectrum and the resolvent in infinite-dimensional space and rewrite this model as an initial value Cauchy Problem before using bounded and Desch perturbation Theorems ($e.g$, see [9, Proposition 11.6] and [45, Theorem0.1]).

First, we consider the following standard definitions and relations. For a given linear operator L with $D(L)$ defined in Banach space X, the spectrum of L is the set of spectral values $\sigma(L) := \{\lambda \in \mathbb{C}; \lambda I - L : D(L) \longrightarrow X$ is not bijective or its inverse is not continuous$\}$. The subset $P\sigma(L) := \{\lambda \in \mathbb{C}; \lambda I - L : D(L) \longrightarrow X$ is not injective$\}$ of $\sigma(L)$ is called the point spectrum of L and consists of eigenvalues. The spectral bound $s(L)$ of L is $sup\{Re\lambda : \lambda \in \sigma(L)\}$ and the peripheral spectrum $\sigma_0(L)$ is $sup\{\lambda_1 \in \sigma(L) : Re\lambda_1 = s(L)\}$.
If L is closed and λ is a spectral value, then the generalized eigenspace $\mathcal{N}_\lambda(L)$ is the smallest closed subspace of X containing $\bigcup_{k=1}^{\infty} N((\lambda I - L)^k)$. The essential spectrum $E\sigma(L)$ is $\{\lambda \in \sigma(L)$ either $(\lambda I - L)(X)$ is not closed, λ is a limit point of $\sigma(L)$, or $\mathcal{N}_\lambda(L))$ is infinite-dimensional$\}$.
If L is bounded, the spectral radius $r_\sigma(L)$ is $sup\{|\lambda| : \lambda \in \sigma(L)\}$, the essential spectral radius $r_{E\sigma}(L)$ is $sup\{|\lambda| : \lambda \in E_\sigma(L)\}$. The resolvent set of L is $\rho(L) := \mathbb{C} \setminus \sigma(L)\ i.e$
$$\rho(L) := \{\lambda \in \mathbb{C}; \lambda I - L : D(L) \longrightarrow X \text{is bijective with continuous inverse}\}.$$

So, for any $\lambda \in \rho(L)$ the operator $\lambda I - L$ has an algebraic inverse called the resolvent operator of L at the point λ denoted it by $R(\lambda, L) := (\lambda I - L)^{-1}$. When L is defined in E, $R(., L)$ is said to be positive if there exists $w \in \mathbb{R}$ such that $]w, +\infty[\subset \rho(L)$ and $R(\lambda, L) \geq 0 \quad \forall\ \lambda > w$. For instance, L generates a positive C_0-semigroup T on E is equivalent to its operator resolvent is positive by the relation

$$R(\lambda, L)x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt,\ \lambda > s(L),\ x \in E.$$

Second, let us introduce the linear operators in E defined as follows: $B_1(u, N) = (B_0 u, -\mu N)$, $B_2(u, N) = (0_{l^1(\mathbb{Z}_+)}, \mathcal{F}u)$, $L_1(u, N) = (L_0 u, 0)$ and
$L_2(u, N) = (\beta p N, 0_{l^1(\mathbb{Z}_+^*)}) = N((\beta p \delta_{n,0})_{n\in\mathbb{Z}_+}, 0)$, where $B_0 u = (a_n u_n)_{n\in\mathbb{Z}_+}$,
$L_0 u = (0, (\gamma_{n-1}u_{n-1})_{n\in\mathbb{Z}_+^*})$ and $\mathcal{F}u = \sum_{n=1}^{+\infty} \kappa_n u_n$, for all $n \in \mathbb{Z}_+$ with $a_n = -(\alpha_n + \kappa_n + \gamma_n)$ and $\delta_{n,0} = 1$ if $n = 0$ else $\delta_{n,0} = 0$ ($\delta_{n,0}$ is a Kronecker symbol). When assumptions (H1) (H2) and (H3) are satisfied, It follows that the domains of operators B_0, B_1, B_2, L_0, L_1, L_2 and $\mathcal{F}$ are $D(B_0) := \{(u_n)_{n\in\mathbb{Z}_+} \in l^1(\mathbb{Z}_+) : (\kappa_n u_n)_{n\in\mathbb{Z}_+} \in l^1(\mathbb{Z}_+))\}$, $D(B_1) = D(B_0) \times \mathbb{R}$,
$D(B_2) = D(B_1)$, for $i = 0, 1, 2$ $D(L_i) = l^1(\mathbb{Z}_+) \times \mathbb{R}$ and $D(\mathcal{F}) = D(B_0)$ respectively. Besides, the linear operators B_0, B_1, B_2 and $\mathcal{F}$ are unbounded in E contrary to L_1 and L_2.
Note that under assumptions (H1) and (H2), the domain of the operator B_0 becomes
$D(B_0) = \{(u_n)_{n\in\mathbb{Z}_+} \in l^1(\mathbb{Z}_+) : (a_n u_n)_{n\in\mathbb{Z}_+}\}$.
Now, let us set $A = B + L$ where $B = B_1 + B_2$ and $L = L_1 + L_2$. Using these notations, it follows that the set of equations (1) can be rewritten in the $AL-space$ E as the following abstract initial value problem:

$$\begin{cases} \frac{dU(t)}{dt} &= AU(t),\ t > 0, \\ U(0) &= x. \end{cases}$$

where $x = (u_0, N_0) \in D(A) \cap E_+$ and for all $t \geq 0 \quad U(t) = (u(t), N(t))$. Thus, to prove that the model (1) is mathematically and ecologically well-posed in E, we only need to show that the operator A with domain $D(A) = D(B_0) \times \mathbb{R} \subseteq l^1(\mathbb{Z}_+)) \times \mathbb{R}$, generates a strongly continuous positive semigroup in E.

Lemma II.1:
The operator B is the generator of a positive C_0-semigroup.

Proof. It's obvious that B_1 is a multiplication operator so that, from (H1) and positivity of all parameters α_n, γ_n, κ_n and β (see (H1)-(H3)), it holds

$$\forall\ n \in \mathbb{Z}_+ \quad a_n < -\check{\alpha}.$$

Therefore, from [9, Proposition.9.21], B_1 is the generator of the infinitesimal (precisely analytic) positive C_0-semigroup

$$(T_{B_1}(t)(u, N))_{t\geq 0} := ((e^{\,a_n t} u_n)_{n\in\mathbb{Z}_+}, e^{-\mu t}N))_{t\geq 0}$$

satisfying

$$\|T_{B_1}(t)\| \leq e^{-\min(\check{\alpha},\mu)t}.$$

Thus, $T_{B_1}(t) \geq 0$ implies $R(\lambda, B_1) \geq 0$ for $\lambda > s(B_1)$.
But, B_2 is a positive operator. So, to obtain the expected result, it is sufficient to show that $B = B_1 + B_2$ is positive operator resolvent. If this holds, it follows from Desch's theorem [45, theorem 0.1] that $B_1 + B_2$ is the generator of a positive C_0-semigroup.

Now, we are going to establish that B is resolvent positive. Since B_1 is resolvent positive operator and B_2 is a positive operator on its domain $D(B_1)$ a subspace of the AL-space E, by [45, Theorem 1.1], sufficient to show that $r_\sigma(B_2 R(\lambda, B_1)) < 1$ for $\lambda > s(B_1)$.
So, let $\lambda > s(B_1)$. From the following spectral radius properties

$$r_\sigma(B_2 R(\lambda, B_1)) = \lim_{k\to+\infty} \|(B_2 R(\lambda, B_1))^k\|^{\frac{1}{k}} = \lim_{k\to+\infty} \|B_2^k (R(\lambda, B_1))^k\|^{\frac{1}{k}}.$$

and

$$B_2^2(u, N) = B_2(0_{l^1(\mathbb{Z}_+)}, \mathcal{F}u) \Longrightarrow B_2^2(u, N) = (0_{l^1(\mathbb{Z}_+)}, 0),$$

we thus obtain $r_\sigma(B_2 R(\lambda, B_1)) = 0 < 1$. □

Lemma II.2:
L is a linear, bounded and positive operator on E so that

$$\|L\|_{\mathcal{L}(E)} \leq \max(\beta p, \hat{\gamma}).$$

Proof. Note that

$$\|L(u, N)\|_E = |\beta p N| + \sum_{n=1}^{+\infty} |\gamma_{n-1} u_{n-1}|$$

Then from (H2), we obtain

$$\begin{aligned} \|L(u, N)\|_E &\leq \beta p|N| + \hat{\gamma}\|u\|_{l^1(\mathbb{Z}_+)} \\ \|L(u, N)\|_E &\leq \max(\beta p, \hat{\gamma})(|N| + \|u\|_{l^1(\mathbb{Z}_+)}) \\ \|L(u, N)\|_E &\leq \max(\beta p, \hat{\gamma})\|(u, N)\|_E. \end{aligned}$$

Hence

$$\|L\|_{\mathcal{L}(E)} \leq \max(\beta p, \hat{\gamma}).$$

□

Finally, by the lemma II.1, lemma II.2 and [9, Corollary 11.7], $A = B + L$ is then the generator of a positive C_0-semigroup denoted it by $(T_A(t))_{t\geq 0}$.

III BALANCED EXPONENTIAL GROWTH OF SOLUTIONS

In this section, we give the basic reproduction number formula and show which ranges of its values induce balanced exponential growth and null asymptotic behavior of solutions exclusively.

Notice that, for the C_0 semigroup $(T_A(t))_{t\geq 0}$ generated by A, the growth bound and the essential type of A defined as

$$w_0(A) := \inf\{\omega \in \mathbb{R}, \exists M_\omega > 1 : \|S(t)\| \leq M_\omega e^{\omega t}, \ \forall t \geq 0\}$$

and

$$w_1(A) := \inf\{\omega \in \mathbb{R}, \exists M_\omega > 1 : \inf\{\|T_A(t) - K\| : \ K \text{ is compact}\} \leq M_\omega e^{\omega t}, \ \forall t \geq 0\}$$

respectively exist and it holds the following identities:

$$\omega_0(A) = \lim_{t\to\infty} \frac{\log(\|T_A(t)\|)}{t},$$

$$\omega_1(A) = \lim_{t\to\infty} \frac{\log(\theta[T_A(t)])}{t}$$

and

$$\omega_0(A) = \max\{\omega_1(A)\ ,\ \sup_{\lambda\in\sigma(A)-E_\sigma(A)} Re\lambda\}$$

where $\theta[L] = inf_{\epsilon>0}\{L(\mathbb{B})$ can be covered by a finite number of balls of radius $\leq \epsilon\}$, is a Hausdorff measure of non compactness ($e.g.$, see [7, 8] for details) satisfying $\theta(L) = 0$ if L is a compact operator and $\mathbb{B}$ is the unit ball of E.
Moreover, $(T_A(t))_{t\geq 0}$ is said to be irreducible if $\forall\, U \in E, \forall\, W \in E^*$ (the linear and topological dual of E), $U > 0$, $W > 0$, we have that $< T_A(t_0)U, W >> 0$ for some $t_0 > 0$, where $< .,. >$ denotes the dual product between E and E^* (see [6, C-III, Definition 3.1]). In order to use these relations in the following, we first decompose the semigroup $(T_A(t))_{t\geq 0}$ in the next theorem, show thereafter that $(T_A(t))_{t\geq 0}$ is quasi-compact ($i.e,\ w_1(A) < 0$) and thereafter give $\mathcal{R}_0$ formula from the characteristic equation before giving its relations with the asymptotic behavior of model (1) solutions.

Theorem III.1:
If assumptions (H1), (H2), (H3) and (H4) are satisfied, then there exist a positive C_0-semigroup, $(T_C(t))_{t\geq 0}$ and a family of compact operators $(V(t))_{t\geq 0}$ such that the positive C_0-semigroup, $(T_A(t))_{t\geq 0}$ generated by A is written as

$$T_A(t)x = T_C(t)x + V(t)x \qquad \forall\, t \geq 0 \qquad \forall\, x \in E$$

where

$$\|T_C(t)\| \leq e^{-\breve{\alpha} t}$$

Proof. Here, we consider the following decomposition of A :

$$A = C + L_3$$

where $C = A - L_3$ and L_3 is defined by

$$L_3(u, N) = N(\beta p(\delta_{n,0})_{n\in\mathbb{Z}_+}, \breve{\alpha} - \mu).$$

Note that L_3 is a compact operator because rank one and bounded linear operator. Also, Desch's theorem (see [45, Theorem 0.1]) and bounded perturbations theorem (see [9, Proposition 11.6]) of the operator B_1 give that C generates a positive C_0-semigroup, denoted by $(T_C(t))_{t\geq 0}$. This semigroup is related to the positive semigroup generated by A, $(T_A(t))_{t\geq 0}$, by this equation

$$T_A(t)x = T_C(t)x + V(t)x, \ \ t \geq 0, \qquad x \in E \tag{8}$$

where

$$V(t) = \int_0^t T_C(t-s)L_3 T_A(s)ds. \tag{9}$$

Since L_3 is compact and $T_A(t) \in \mathcal{L}(E)$ for all $t \geq 0$, the map $t \mapsto L_3 T_A(t)$ is then compact. Thus, for the same reasons, it results $V(t)$ is a compact operator for all $t \geq 0$.

Now, we show that

$$\|T_C(t)\| \leq e^{-\breve{\alpha} t} \qquad \forall t \geq 0.$$

Let $x \in E_+$, $t \geq 0$ and $U(t) = (u(t), N(t)) \in E_+$ so that $U(0) = x$.

$$\frac{dU(t)}{dt} = CU(t) \Longleftrightarrow \begin{cases} \frac{du_0}{dt} &= -(\alpha_0 + \gamma_0) u_0, \\ \frac{du_n}{dt} &= a_n u_n + \gamma_{n-1} u_{n-1}, \ for \ n \geq 1, \\ \frac{dN}{dt} &= -\breve{\alpha} N + \mathcal{F}(u). \end{cases}$$

For a fixed $m \in \mathbb{Z}_+^*$, we consider the m-states system with $\gamma_m = 0$ without loss generality and set

$$S_m(t) = \sum_{n=0}^{m} \frac{du_n(t)}{dt} \text{ and } S(t) = \sum_{n=0}^{+\infty} \frac{du_n(t)}{dt}, \text{ for all } t \geq 0.$$

Therefore, it holds

$$S_m(t) = -\sum_{n=0}^{m} (\alpha_n + \kappa_n) u_n(t) \leq 0.$$

Thus, the sequence of continuous functions $(S_m(t))_{m \in \mathbb{Z}_+}$ is convergent to $S(t)$ which is continuous on $[0, +\infty[$. This convergence is uniform on any compact set of $[0, +\infty[$. Using (H1) and (H3), it results

$$\frac{d}{dt}\|U(t)\| \leq -\breve{\alpha}\|U(t)\|, \text{ for all } t \geq 0,$$

It follows that

$$\|U(t)\| \leq e^{-\breve{\alpha} t}\|U(0)\|, \text{ for all } t \geq 0.$$

So, we have

$$\forall x \in E_+ \quad \|T_C(t)x\| \leq e^{-\breve{\alpha} t}\|x\|, \text{ for all } t \geq 0 \tag{10}$$

Now, let $x \in E$. The properties (5) imply that

$$\|T_C(t)x\| \leq \|T_C(t)x_+\| + \|T_C(t)x_-\| \text{ with } x = x_+ - x_- \text{and } x_+, x_- \in E_+$$

Using again (5), we thus obtain this inequality

$$\|T_C(t)x_+\| + \|T_C(t)x_-\| \leq e^{-\breve{\alpha} t}(\|x_+\| + \|x_+\|)$$

which implies from (6) the result

$$\forall x \in E \quad \|T_C(t)x\| \leq e^{-\breve{\alpha} t}\|x\|, \text{ for all } t \geq 0.$$

□

This Theorem III.1 implies the following lemma about the quasi-compactness of $(T_A(t))_{t \geq 0}$ (see [21, Chapter V, Definition 3.4]).

Lemma III.2:
The semigroup $(T_A(t))_{t \geq 0}$ is quasi-compact such that $w_1(A) \leq -\breve{\alpha}$.

Proof. From Theorem III.1, obvious that hypotheses of [47, Proposition 2.4] hold. Therefore, it happens $w_1(A) \leq -\breve{\alpha} < 0$, by [21, Chapter V, Proposition 3.5], the semigroup $(T_A(t))_{t\geq 0}$ is quasi-compact. □

In the following lemma, we give the basic reproduction number formula which will be, in the proof of the Theorem I.2, associated to the spectral bound $s(A)$ of the operator A.

Lemma III.3:
Let a real λ be in the point spectrum of A. Then, basic reproduction ratio of the model (1) is given by

$$\mathcal{R}_0 := f(0) = \frac{\beta p}{\mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{\alpha_n + \kappa_n + \gamma_n} \prod_{i=0}^{n-1} \frac{\gamma_i}{\alpha_i + \kappa_i + \gamma_i}, \tag{11}$$

and the characteristic function associated to A

$$f(\lambda) = \frac{\beta p}{\lambda + \mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{\lambda + \alpha_n + \kappa_n + \gamma_n} \prod_{i=0}^{n-1} \frac{\gamma_i}{\lambda + \alpha_i + \kappa_i + \gamma_i}.$$

satisfies the following assertions:
a) $\lambda < 0 \iff \mathcal{R}_0 < 1$;
b) $\lambda > 0 \iff \mathcal{R}_0 > 1$;
c) $\lambda = 0 \iff \mathcal{R}_0 = 1$.

Proof. Let λ be a eigenvalue of A, then it exists $v \neq 0_E$ so that $Av = \lambda v$. Consequently, it results from little algebra calculations the characteristic equation is $f(\lambda) = 1$ where the characteristic function f is defined by

$$f(\lambda) = \frac{\beta p}{\lambda + \mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{\lambda + \alpha_n + \kappa_n + \gamma_n} \prod_{i=0}^{n-1} \frac{\gamma_i}{\lambda + \alpha_i + \kappa_i + \gamma_i}.$$

Since the real valued function $\lambda \mapsto f(\lambda)$ is strictly decreasing, we obtain assertions $a)$, $b)$, and $c)$ whenever we set $\mathcal{R}_0 := f(0)$. □

We give the proof of the Theorem I.2

Proof. First, we prove that $(T_A(t))_{t\geq 0}$ is irreducible $i.e$ it exists $t_0 > 0$ such that

$$< T(t_0)U, U^* >> 0 \; .$$

Let A be rewritten as $A = G + H$ with

$$G(u, N) = (B_0 u, -\mu N), \text{ and } H(u, N) = (L_0 u + (\beta p N \delta_{n,0})_{n\in\mathbb{Z}_+}, \mathcal{F}u).$$

Therefore, we have the multiplication operator G is generator of a positive C_0-semigroup. Then, from $A = H + G$ is positive resolvent and H is a positive operator, it holds for any $\lambda > s(G)$

$$R(\lambda, A) = R(\lambda, G) \sum_{n=0}^{\infty} (HR(\lambda, G))^n.$$

To prove that $(T_A(t))_{t\geq 0}$ is irreducible, sufficient to show that for any $U > 0$, $HR(\lambda, G)U > 0$ for some λ. In fact, even if $\lambda > \max(-\breve{\alpha}, -\mu)$ it holds:

$$R(\lambda, G)(u, N) = \left(\left(\frac{u_n}{\lambda + \alpha_n + \kappa_n + \gamma_n}\right)_{n\in\mathbb{Z}_+}, \frac{N}{\lambda + \mu}\right)$$

then

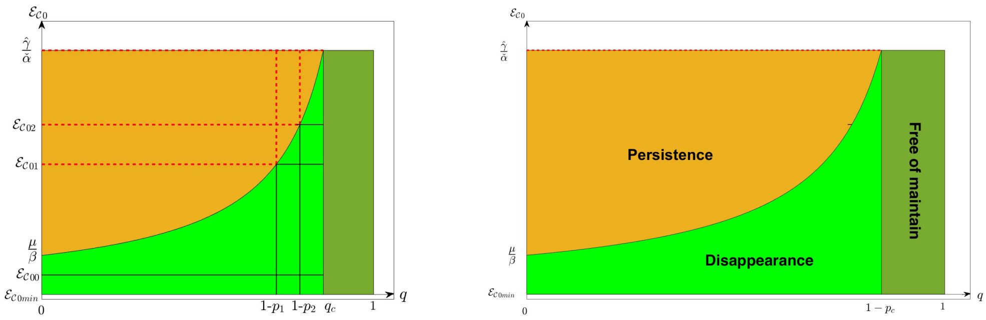
$$HR(\lambda, G)(u, N) = \left(\frac{\beta p N}{\lambda + \mu}, \left(\frac{\gamma_{n-1}}{\lambda + \alpha_{n-1} + \kappa_{n-1} + \gamma_{n-1}} u_{n-1}\right)_{n\geq 1}, \sum_{n=1}^{+\infty} \frac{\kappa_n}{\lambda + \alpha_n + \kappa_n + \gamma_n} u_n\right).$$

So, from applying $HR(\lambda, G)$ to each element of the canonical basis of E, it holds for any $\lambda > \max(-\breve{\alpha}, -\mu)$ and $U > 0$: $HR(\lambda, G)U > 0$. Therefore, we obtain for any $U > 0$ and $U^* > 0$, $< R(\lambda, A)U, U^* >= \int_0^{\infty} e^{-\lambda s} < T_A(s)U, U^* > ds > 0$ for $\lambda > \max(-\breve{\alpha}, -\mu)$ it yields the claim.
Next, we prove i) and ii) respectively.
Before, let us establish two useful results and a remark. First, by [9, Theorem 12.17], A is the generator of the positive C_0-semigroup $(T_A(t))_{t\geq 0}$ on the AL-space E implies $s(A) = w_0(A)$. Second, since $(T_A(t))_{t\geq 0}$ is positive and irreducible in the Banach Lattice $E = l^1(\mathbb{Z}_+) \times \mathbb{R}$, it results, from [6, C.III, Theorem 3.7], that the spectrum $\sigma(A)$ is not empty. So, it follows $s(A) > -\infty$. Therefore, since $(T_A(t))_{t\geq 0}$ satisfies hypotheses of [21, Chapter VI, 1.10 Theorem], on the AL space E, we get $s(A)$ is in the spectrum of A.
Thirty, note that only the result in i) holds (with $s(A) < 0$) whenever $w_0(A) = w_1(A)$. In fact, from the inequalities $\omega_1(A) \leq -\breve{\alpha} < 0$ established in Lemma III.2 and $w_0(A) = s(A)$, it results $s(A) < 0$. So, it holds $Re(\lambda) < 0$ for any $\lambda \in \sigma(A)$ and $\exists\, \varepsilon > 0,\ M \geq 1 :\ \|T_A(t)\| \leq Me^{-\varepsilon t}$.
In the following, we suppose that $\omega_1(A) < w_0(A)$. This inequality implies from [47, Proposition 2.5] that $\lambda_0 = s(A)$ is in the point spectrum of A and $\sigma_0(A) = \{s(A)\}$.
(i) Assume that $\mathcal{R}_0 < 1$. Since $s(A)$ is in the point spectrum of A, from Lemma III.3, it results $s(A) = w_0(A) < 0$ it yields the claim.
(ii) If $\mathcal{R}_0 \geq 1$, from Lemma III.3, it holds $\lambda_0 = s(A) \geq 0$. Therefore, $(T_A(t))_{t\geq 0}$ is quasi-compact and $s(A) \geq 0$ provide (see [21, Chapter V, 3.7 Theorem]) $s(A)$ is a pole of $R(., A)$, the resolvent of A. But, $(T_A(t))_{t\geq 0}$ is irreducible on the AL-space E, then from applying [9, Proposition 14.12] and [47, Proposition 2.3] consecutively, it yields the claim. □

IV ILLUSTRATIVE ASSESSMENT OF ARBOVIRUSES MAINTENANCE IN ENVIRONMENT VIA *AEDES*

In determining how persist arboviruses in ecosystems by the way of transovarial transmission, necessary to know what's happened in a crosscut impact factors and floodwater *Aedes* demographic ones. From the Lemma III.3, the impact factors and egg-Adult demographic parameters in interest here are both directly related to the basic Reproduction rate number $\mathcal{R}_0$ (11) of the growth model (1). Specifically, the $\mathcal{R}_0$ is explicitly the ratio of adult females parameters (β/μ) times the transovarial transmission probability (p) times a factor denoted by $\mathcal{E}_{\mathcal{C}0}$ as we define Eggs Climate-Environment Reactivity Index

$$\mathcal{E}_{\mathcal{C}0} := \sum_{n=1}^{+\infty} \frac{\kappa_n}{\alpha_n + \kappa_n + \gamma_n} \prod_{i=0}^{n-1} \frac{\gamma_i}{\alpha_i + \kappa_i + \gamma_i}. \tag{12}$$



(a) Types of cross lines from q and $\mathcal{E}_{\mathcal{C}0}$ axes: disappearance (solid cross solid) and persistence (crossing curve or outside of solid cross solid).

(b) Areas of washing arbovirus (right) and potential maintenance (left) from crossing lines typology: persistence and disappearance.

Figure 1: Maintain of arbovirus Topography based on a complete map of relations between parameters μ/β (the inverse of a specific *Aedes* adult female oviposition rate), p (transovarial transmission probability) and $\mathcal{E}_{\mathcal{C}0}$ (Eggs Climate-Environment Reactivity Index) in its admissible interval $[\mathcal{E}_{\mathcal{C}0min}; \hat{\gamma}/\breve{\alpha}]$ where $\mathcal{E}_{\mathcal{C}0min}$, $\hat{\gamma}$ and $1/\breve{\alpha}$ represent the minimum of $\mathcal{E}_{\mathcal{C}0}$, the maximal flooding frequency and the long lifetime expectancy of eggs in their breeding habitat respectively. The line $q = q_c$ ($q_c = 1 - p_c$ representing the complementary probability of, p_c, critical transovarial transmission probability) marks separation of free area of arbovirus maintain via eggs and the potential zone of persistence. The Illustrative cases of unconditional disappearance with $\mathcal{E}_{\mathcal{C}00}$ and conditioned persistence by the minimal transovarial transmission probability $p_i = 1 - q_i$ with $\mathcal{E}_{\mathcal{C}0i}$ $(i = 1, 2)$.

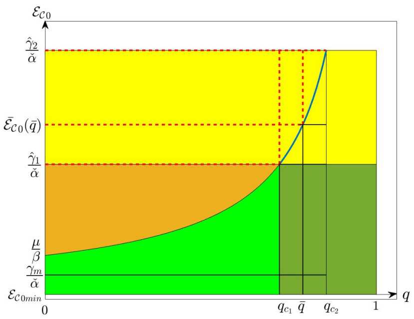
This index contains only eggs parameters of the model affected by environment and climate factors variations. Note that $\mathcal{E}_{\mathcal{C}0}$ is not only dimensionless but is less than $\hat{\gamma}/\breve{\alpha}$. Also, when the transovarial transmission is inhibited ($p = 0$), it obviously holds $\mathcal{R}_0 = 0$, then the model (1) converge to zero by assertion i) of the Theorem I.2. This confirms that when $p = 0$ the system is always free of considered arbovirus under any climate and environment variations of the breeding habitat.

The situation of interest here is when transovarial transmission exists and is not inhibited ($0 < p \leq 1$). In this case, if $\mathcal{R}_0$ is unity ($\mathcal{R}_0 = 1$), the Index $\mathcal{E}_{\mathcal{C}0}$ gives a specific *Aedes spp*-arbovirus critical value, $\mathcal{E}_{\mathcal{C}0}$, of the Eggs Climate-Environment Reactivity Index defined as

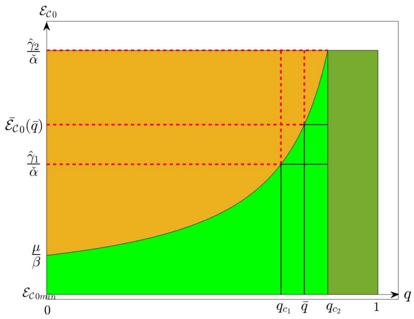
$$\mathcal{E}_{\mathcal{C}0} = \frac{\mu}{\beta} \times \frac{1}{1-q} \tag{13}$$

where q is the complementary probability of p ($i.e.$ $p + q = 1$). Therefore, when p varies the expression of $\mathcal{E}_{\mathcal{C}0}$ becomes an increasing function of q denoted by $\bar{\mathcal{E}}_{\mathcal{C}0}(q)$, from $\mathcal{E}_{\mathcal{C}0} \leq \hat{\gamma}/\breve{\alpha}$, this function, $\bar{\mathcal{E}}_{\mathcal{C}0}(q)$, is upper limited by $\bar{\mathcal{E}}_{\mathcal{C}0}(q_c)$ where $q_c = 1 - (\breve{\alpha}\mu)/(\hat{\gamma}\beta)$. From the Theorem I.2, it hence holds that for a fixed q or p, the quantity $\bar{\mathcal{E}}_{\mathcal{C}}(q)$ is a bifurcation value of the model (1) with respect to $\mathcal{E}_{\mathcal{C}0}$ according to its relation with $\mathcal{R}_0$.

For instance, when $p = 1$ it holds that the critical value, $\mathcal{E}_{\mathcal{C}0}(0)$, of the Index $\mathcal{E}_{\mathcal{C}0}$ is μ/β. So, if $\mathcal{E}_{\mathcal{C}0} < \mu/\beta$ holds then the model (1) converges to zero ($i.e$, the arbovirus disappear in the habitat), else it converges to non null projector ($i.e$, the arbovirus persists in the system by the transovarial transmission way), by analogy for each p such that $p_c < p \leq 1$, it results similar conclusions about persistence and disappearance in *Aedes* breeding habitat of any arbovirus subject of transovarial transmission depicted in Figure.1(a) and Figure.1(b).



(a) Conditions of free maintain arbovirus in breeding site 1 via eggs with maximal flooding frequency $\hat{\gamma}_1$ and critical vertical transmission probability, $p_{c_1} = 1 - q_{c_1}$ $(p_{c_1} > p = 1 - \bar{q})$.



(b) Conditions of maintain arbovirus in breeding site 2 via eggs with maximal flooding frequency $\hat{\gamma}_2$ and critical vertical transmission probability, $p_{c_2} = 1 - q_{c_2}$ $(p_{c_2} < p = 1 - \bar{q})$.

Figure 2: Maintain (right) and free of maintain (left) of arbovirus topography statements based on a complete map of relations between model (1) parameters of two different *Aedes* breeding habitats in flooding frequencies $(\hat{\gamma}_1 < \hat{\gamma}_2)$. Transovarial transmission probability $p = 1 - \bar{q}$ $(p_{c_2} < p < p_{c_1})$, total oviposition of specific *Aedes* adult female during its lifetime β/μ and maximal lifetime of eggs $1/\breve{\alpha}$ parameters are the same.

These Figures (1(a) and 1(b)) show that a transmitted arbovirus will die out under any climate-environment conditions in the *Aedes* breeding habitat in interest when the probability of transovarial transmission is less than the quantity $p_c = (\breve{\alpha}\mu)/(\hat{\gamma}\beta)$. Else, it can persist or disappear according to the Index $\mathcal{E}_{C0}$ value. Note that the critical vertical transmission probability, p_c, may vary from an *Aedes* breeding habitat to another as it is depicted for instance in Figures (2(a) and 2(b)), when two *Aedes* breeding habitats (site 1 and site 2) are only different in terms of maximal flooding frequency, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ respectively. It is obvious seen that their critical probabilities satisfy $p_{c_2} < p_{c_1}$ if $\hat{\gamma}_1 < \hat{\gamma}_2$.

Finally, this qualitative analysis from the model (1) reveals that it always exists lowest transovarial transmission rates range on which a considered arbovirus living in a floodwater *Aedes spp* breeding habitat should not be maintained via eggs. But, out of this range, the arbovirus maintenance in a system via *Aedes* vertical transmission depends greatly on specific adult female reproduction rate and $\mathcal{E}_{C0}$ index values.

V CONCLUSION

In this work, we study by positive semigroup approach an infinite dimension ordinary differential equations system describing the transmission of arbovirus between eggs under flooding events and adult females of *Aedes* mosquito by vertical-transovarial infection. Using positive semigroup theory, we proved that the model is mathematically and ecologically well posed, analyzed by spectral theory and perturbations technics its asymptotic behavior and gave an illustrative application in a contributive diagram to the assessment of an arbovirus maintenance in ecosystems by the way of mosquito's transovarial transmission. To extend these results, it would be interesting to model resource-induced competition during mosquito aquatic stages by nonlinear term and study the corresponding semi-linear problem. This perspective might also be considered in a future work.

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VII APENDIX

7.1 The computation of $\mathcal{R}_0$

For all $\lambda > \max(-\breve{\alpha}, -\mu)$, if $\lambda \in \sigma_p(A)$, then there is $v \neq 0_E$ such that $Av = \lambda v$. Thus, let $v := \begin{pmatrix} (u_n)_{n\in\mathbb{Z}_+} \\ N \end{pmatrix}$. Then it derives that:

$$\begin{pmatrix} -(\alpha_0+\gamma_0)u_0 + \beta p N \\ (-(\alpha_n+\kappa_n+\gamma_n)u_n + \gamma_{n-1}u_{n-1})_{n\geq 1} \\ \sum_{n=1}^{+\infty} \kappa_n u_n - \mu N \end{pmatrix} = \begin{pmatrix} \lambda u_0 \\ \lambda (u_n)_{n\geq 1} \\ \lambda N \end{pmatrix}.$$

Hence

$$\begin{cases} (\alpha_0+\gamma_0+\lambda)u_0 & = & \beta p N, \\ (\alpha_n+\kappa_n+\gamma_n+\lambda)u_n & = & \gamma_{n-1}u_{n-1}, \ n \geq 1, \\ (\lambda+\mu)N & = & \sum_{n=1}^{+\infty} \kappa_n u_n. \end{cases}$$

For $i \in \{0, 1, 2......., n\}$, by taking the product of the member-to-member equations, we obtain:

$$\begin{cases} \prod_{i=0}^{n}(\alpha_i+\kappa_i+\gamma_i+\lambda)u_n & = & \beta p N \prod_{i=0}^{n-1}\gamma_i, \\ (\lambda+\mu)N & = & \sum_{n=1}^{+\infty}\kappa_n u_n, \end{cases} \iff \begin{cases} u_n = \beta p N \dfrac{\prod_{i=0}^{n-1}\gamma_i}{\prod_{i=0}^{n}(\alpha_i+\kappa_i+\gamma_i+\lambda)}, \\ (\lambda+\mu)N = \sum_{n=1}^{+\infty}\kappa_n u_n, \end{cases}$$

and therefore

$$
\begin{aligned}
(\lambda+\mu)N &= \sum_{n=1}^{+\infty} \kappa_n \beta p N \frac{\prod_{i=0}^{n-1}\gamma_i}{\prod_{i=0}^{n}(\alpha_i+\kappa_i+\gamma_i+\lambda)} \\
&= \beta p N \sum_{n=1}^{+\infty} \frac{\kappa_n}{(\alpha_n+\kappa_n+\gamma_n+\lambda)} \prod_{i=0}^{n-1} \frac{\gamma_i}{(\alpha_i+\kappa_i+\gamma_i+\lambda)}.
\end{aligned}
$$

Thus, we obtain

$$
\frac{\beta p}{\lambda+\mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{(\lambda+\alpha_n+\kappa_n+\gamma_n)} \prod_{i=0}^{n-1} \frac{\gamma_i}{(\lambda+\alpha_i+\kappa_i+\gamma_i)} = 1.
$$

Let us pose

$$
\mathcal{R}_0 := f(0) = \frac{\beta p}{\mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{(\alpha_n+\kappa_n+\gamma_n)} \prod_{i=0}^{n-1} \frac{\gamma_i}{(\alpha_i+\kappa_i+\gamma_i)},
$$

with

$$
f(\lambda) = \frac{\beta p}{\lambda+\mu} \sum_{n=1}^{+\infty} \frac{\kappa_n}{(\lambda+\alpha_n+\kappa_n+\gamma_n)} \prod_{i=0}^{n-1} \frac{\gamma_i}{(\lambda+\alpha_i+\kappa_i+\gamma_i)}.
$$

Note that $\lambda \mapsto f(\lambda)$ is a decreasing function and so we have the following equivalences:

a) $\lambda < 0 \iff \mathcal{R}_0 < 1$;
b) $\lambda > 0 \iff \mathcal{R}_0 > 1$;
c) $\lambda = 0 \iff \mathcal{R}_0 = 1$.

7.2 Abstract formulation of system (1)

Note that

$$
\frac{dU}{dt} = \begin{pmatrix} (-(\alpha_n+\kappa_n+\gamma_n)u_n + \beta p N \delta_{n,0} + \gamma_{n-1}u_{n-1}.1_{\{n\geq 1\}})_{n\in\mathbb{Z}_+} \\ \sum_{n=1}^{+\infty} \kappa_n u_n - \mu N \end{pmatrix}
$$

and therefore

$$
\frac{dU}{dt} = \begin{pmatrix} B_0 u \\ -\mu N \end{pmatrix} + \begin{pmatrix} 0_{l^1(\mathbb{Z}_+)} \\ \mathcal{F}u \end{pmatrix} + \begin{pmatrix} L_0 u \\ 0 \end{pmatrix} + \begin{pmatrix} (\beta p N \delta_{n,0})_{n\in\mathbb{Z}_+} \\ 0 \end{pmatrix}
$$

with

$$
B_0 u = (-(\alpha_n+\kappa_n+\gamma_n)u_n)_{n\in\mathbb{Z}_+}\ ,\ \mathcal{F}u = \sum_{n=1}^{+\infty} \kappa_n u_n \text{ and } L_0 u = \begin{cases} \gamma_{n-1}u_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}.
$$

Thus, it becomes:

$$
\frac{dU}{dt} = B_1 U + B_2 U + L_1 U + L_2 U,
$$

where

$$B_1 U = \begin{pmatrix} B_0 u \\ -\mu N \end{pmatrix}, \; B_2 U = \begin{pmatrix} 0_{l^1(\mathbb{Z}_+)} \\ \mathcal{F} u \end{pmatrix}, \; L_1 U = \begin{pmatrix} L_0 u \\ 0 \end{pmatrix} \text{ et } L_2 U = \begin{pmatrix} (\beta p N \delta_{n,0})_{n \in \mathbb{Z}_+} \\ 0 \end{pmatrix}.$$

Thus, let us pose

$$A = B + L \text{ with } B = B_1 + B_2 \text{ and } L = L_1 + L_2.$$

Then, one can obtain :

$$\frac{dU}{dt} = AU.$$